APPENDIX A: 2-PARTICLE IRREDUCIBILITY

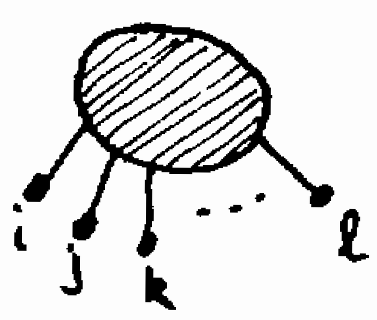
The virtue of the diagrammatic derivation of the 1PI Green functions, section 2.G, is that one does not need to prove 1P-irreducibility; it is built-in, by construction. To test the power of the method, I do it here for 2-particle irreducible Green functions, and am (almost) successful. This is a warming-up exercise for computing QCD bound states. Besides, it is crowding my notebooks.

Introduce <u>2 kinds of sources</u>: $J = (J_i, J_{ij})$

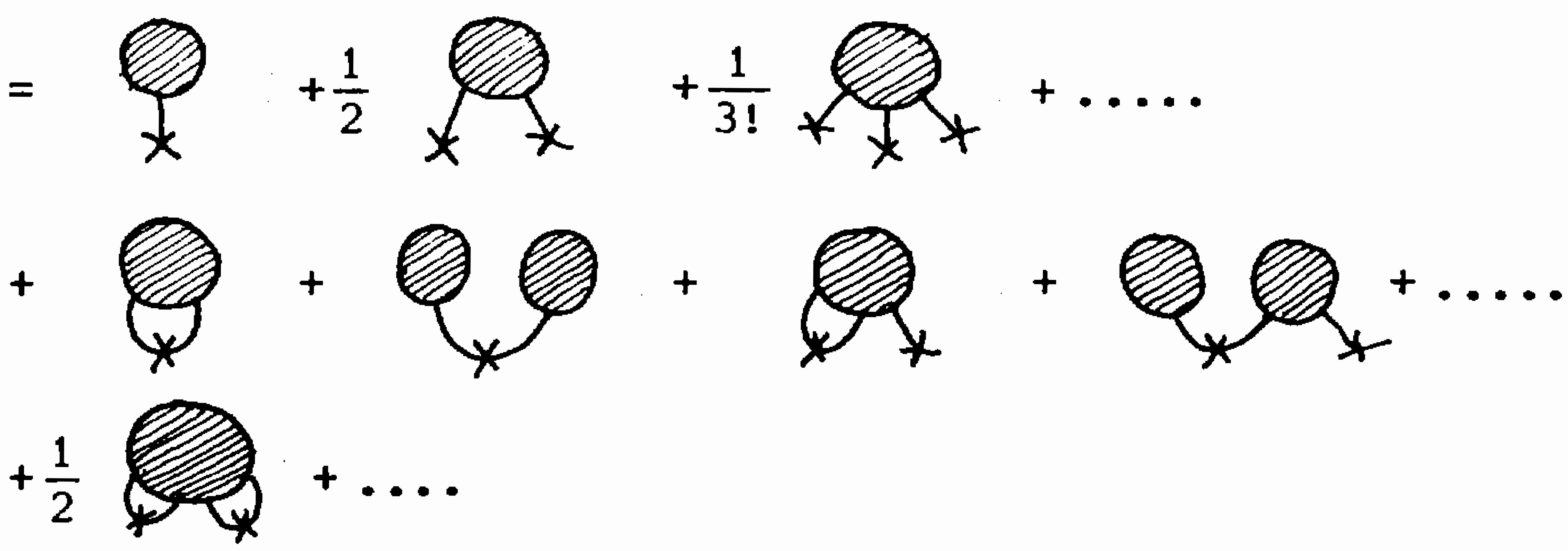
1-particle sources $\quad J_i = $ x——i

2-particle sources $\quad J_{ij} = $ i——x——j $= J_{ji}$ (A.1)

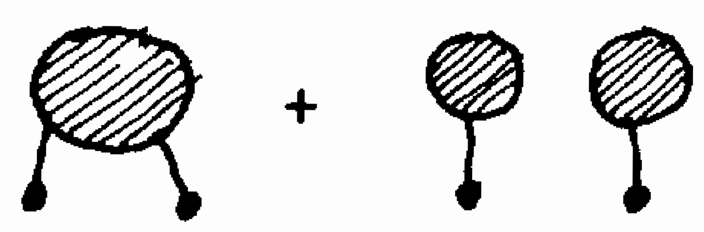
The <u>connected</u> Green functions are the same as usual

$G^{(c)}_{ijk\ldots\ell} = $  $= \frac{d}{dJ_i}\frac{d}{dJ_j}\cdots\frac{d}{dJ_\ell}W[J]\Big|_{J=0}$

as they are evaluated at $J_i = J_{ij} = 0$. The generating functional is a double expansion in J_i and J_{ij};

$W[J] = $  (A.2)

Removing a two-particle source can disconnect a connected diagram:

$\frac{dW[J]}{dJ_{ij}} = $  $= \frac{d^2W[J]}{dJ_i dJ_j} + \frac{dW[J]}{dJ_i}\frac{dW[J]}{dJ_j}$ (A.3)

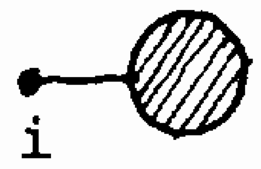
Nota bene:

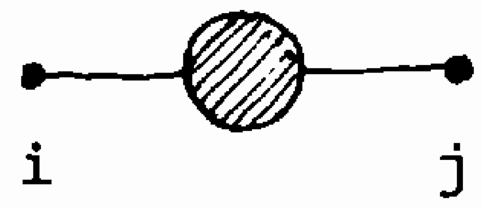
$$\frac{d}{dJ_{ij}} J_{mn} = \frac{1}{2}\left(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}\right),\tag{A.4}$$

do not forget symmetrizations!

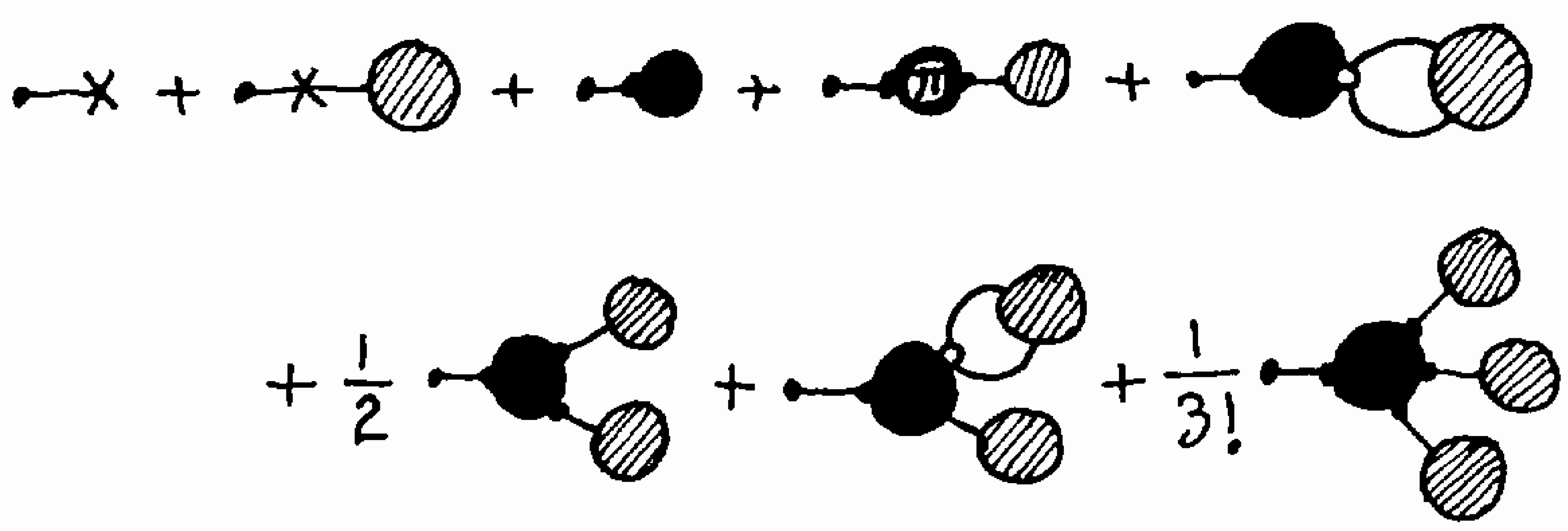
To define 2-particle irreducible graphs, we have to remove tadpoles (connected to the rest of the diagram by 1 line) and self-energy insertions (connected to the rest of the diagram by 2 lines), Hence introduce

$$\phi = \left(\phi_i, D_{ij}\right)$$

fields: $\phi_i = \frac{dW[J]}{dJ_i} =$ 

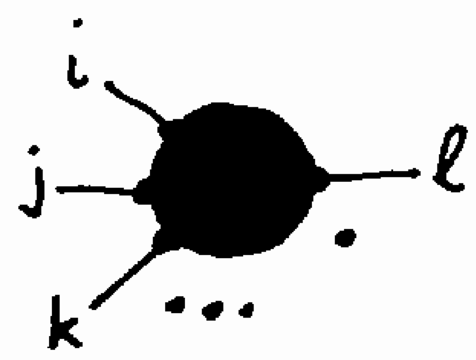
propagators: $D_{ij} = \frac{d^2W[J]}{dJ_i dJ_j} =$  (A.5)

If we pull out a leg, it either ends on a source, or 2PI diagram, or 2P-reducible diagram:

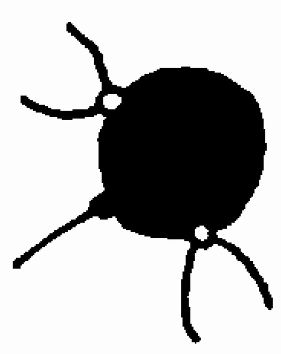
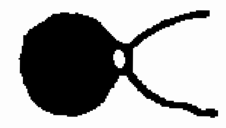
$\phi_i = \frac{dW[\vec{J}]}{dJ_i} =$  (A.6)

$$\phi_i = \Delta_{ij}\Big(J_j + J_{jk}\phi_k + \Gamma_j + \pi_{jk}\phi_k + \Gamma_{j\underline{ki}}D_{k\ell}$$
$$+ \frac{1}{2}\Gamma_{jk\ell}\phi_k\phi_\ell + \Gamma_{jk\underline{\ell m}}\phi_k D_{\ell m} + \frac{1}{3!}\Gamma_{jk\ell m}\phi_k\phi_\ell\phi_m + \ldots\Big) .$$

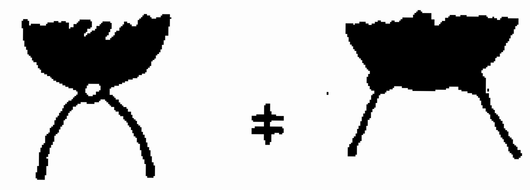
The <u>2-particle irreducible</u> (2PI) Green functions are drawn as black blobs, with each external line coming into a separate vertex:

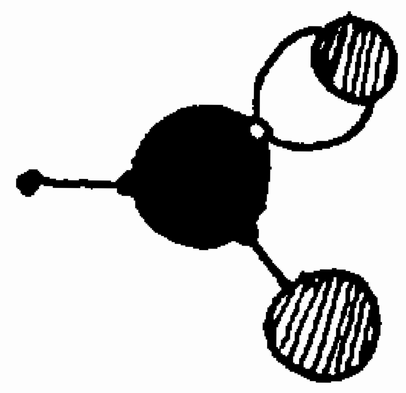
$\Gamma_{ijk..\ell} =$  $= \frac{d}{d\phi_i}\frac{d}{d\phi_j}\frac{d}{d\phi_k}\cdot\cdot\frac{d}{d\phi_\ell}\Gamma[\phi]\Big|_{\phi_i = D_{ij} = 0}$. (A.7)

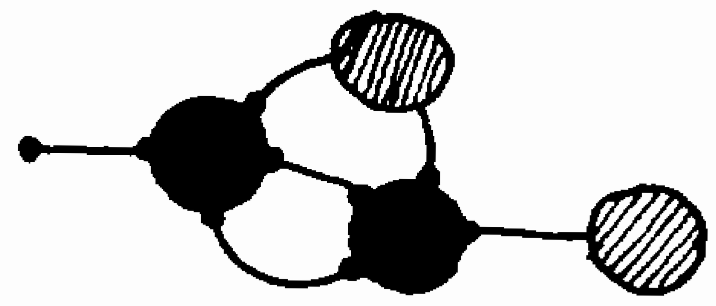
Derivatives with respect to self-energies are denoted by the corresponding pairs of lines coming into a white vertex:

$\Gamma_{\underline{ij}k\underline{\ell m}..} =$  $= \frac{d}{dD_{ij}}\frac{d}{d\phi_k}\frac{d}{dD_{\ell m}}\Gamma[\phi]\Big|_{\phi=0}$ (A.8) caution: 1)  can be 2-particle reducible

2) $\frac{d}{dD_{ij}} \neq \frac{d}{d\phi_i}\frac{d}{d\phi_j}$

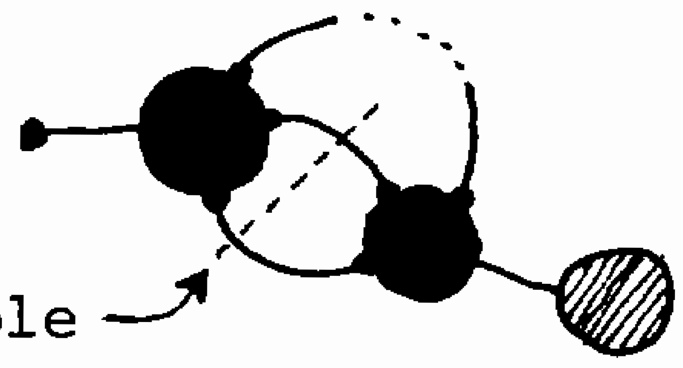


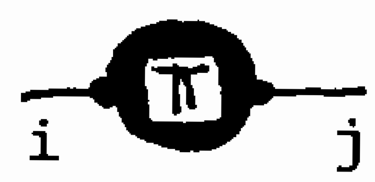
example: a term like  contains diagrams such as



When

when we remove the propagator, the remainder is 2-particle reducible

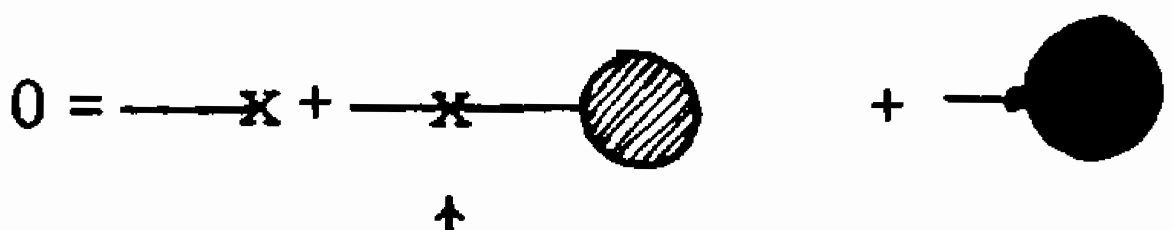


In the above expansion of dW/dJ_i, the $\pi_{ij} =$  term is 2PI. We sum up its iteration by defining

$$\Gamma_{ij} = -\Delta^{-1}_{ij} + \pi_{ij} \quad , \tag{A.9}$$

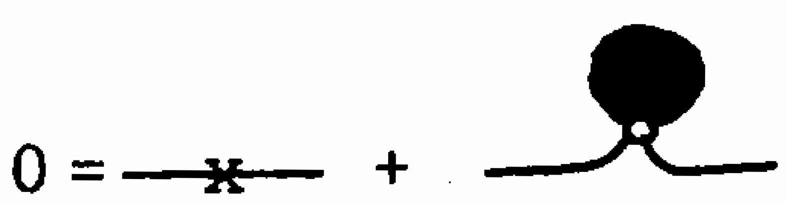
and the expansion can be rewritten as the first duality relation:

$$0 = J_j + J_{jk}\phi_k + \frac{d\Gamma[\phi]}{d\phi_j}$$

 (A.10)

The second duality relation is

$$0 = J_{ij} + \frac{d\Gamma[\phi]}{dD_{ij}}$$

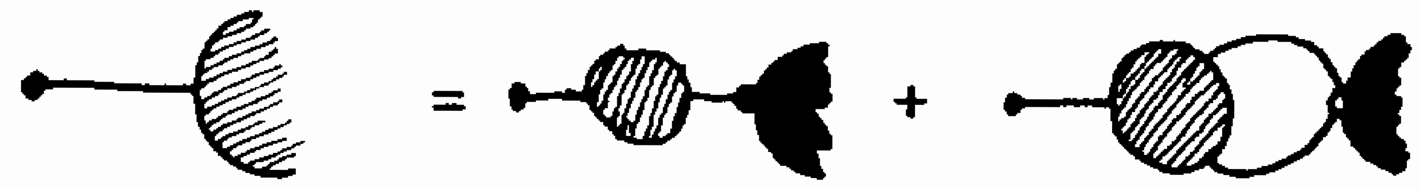
 (A.11)

I do not know how to derive this diagrammatically[†], but algebraically it comes from the <u>second Legendre transform</u>:

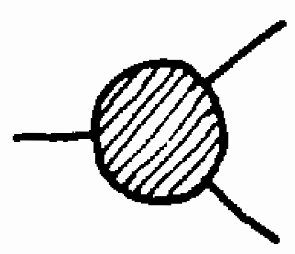
$$\Gamma[\phi] = W[J] - \frac{dW[J]}{dJ_i} J_i - \frac{dW[J]}{dJ_{ij}} J_{ij} \,, \tag{A.12}$$

by differentiating with respect to D_{ij}. To go from connected to 2PI Green functions, use the chain rule:

$$\frac{d}{dJ_i} = \frac{d\phi_j}{dJ_i}\frac{d}{d\phi_j} + \frac{dD_{jk}}{dJ_i}\frac{d}{dD_{jk}} = D_{ij}\frac{d}{d\phi_j} + \frac{d^3W[J]}{dJ_i dJ_j dJ_k}\frac{d}{dD_{jk}} \,, \tag{A.13}$$



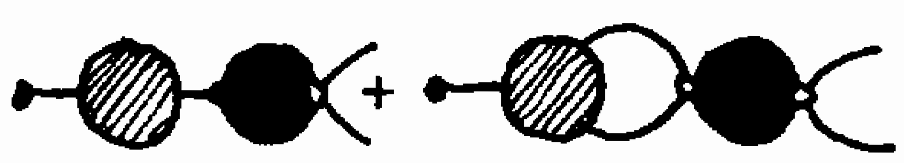
†this has to be re-expressed in terms of ϕ_i, D_{ij}

To eliminate , use the identity

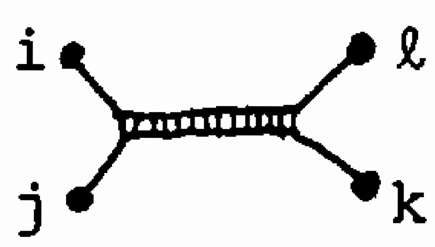
$$\frac{dJ_{mn}}{dJ_i} = 0 \,. \tag{A.14}$$

Substituting $d\Gamma/dD_{mn}$ for J_{mn} and using the chain rule, we obtain

$$0 = \left(D_{ij}\frac{d}{d\phi_j} + \frac{d^3W[J]}{dJ_i dJ_j dJ_k}\frac{d}{dD_{jk}}\right)\frac{d\Gamma[\phi]}{dD_{mn}}$$

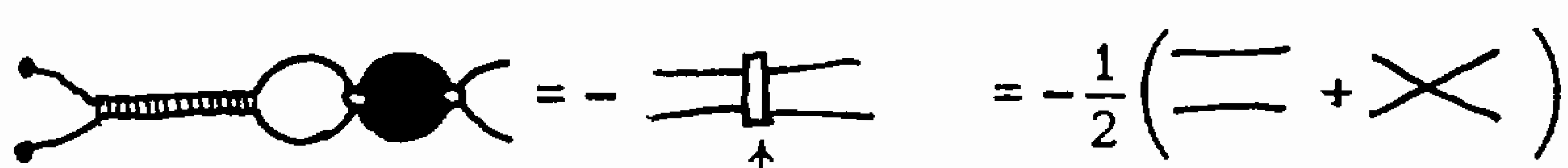
0 =  (A.15)

Define 2-particle propagator as the inverse of $\Gamma_{\underline{k\ell}\,\underline{mn}}$:

$D_{\underline{ij}\,\underline{k\ell}}$ = 

(A.16)

$$D_{\underline{ij}\,\underline{k\ell}}\Gamma_{\underline{k\ell}\,\underline{mn}} = -\frac{1}{2}\left(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}\right)$$


(A.17)

†symmetrized, 2-particle subspace

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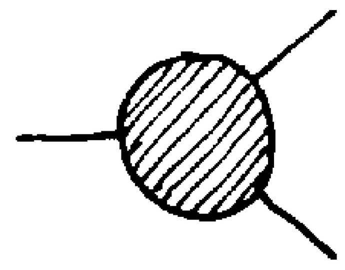
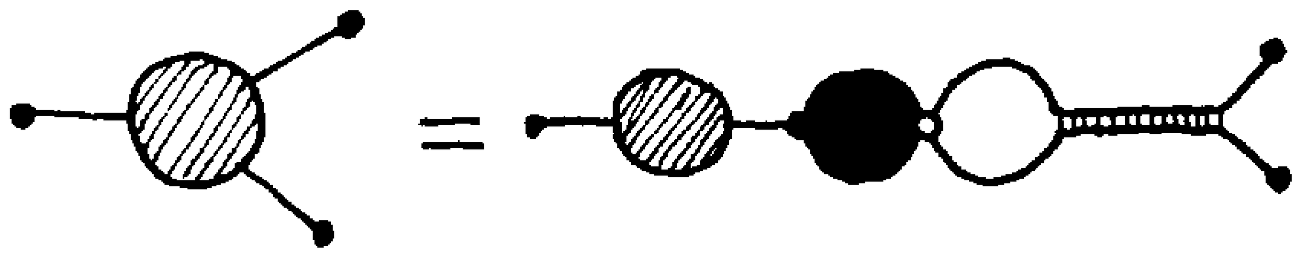
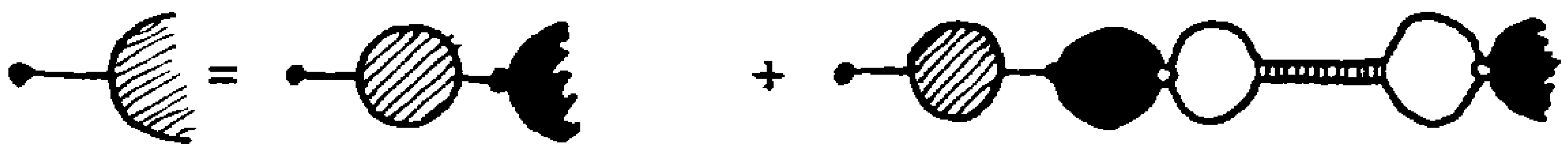
† Here is where my derivation falls flat on its face.

Now we can eliminate  in terms of ϕ_i, D_{ij} functions:

 (A.18)

and the chain rule takes a sensible form

$$\frac{d}{dJ_i} = D_{ij}\left(\frac{d}{d\phi_j} + \frac{d^2\Gamma[\phi]}{d\phi_j dD_{k\ell}} D[\phi]_{\underline{k\ell}\ \underline{mn}} \frac{d}{dD_{mn}}\right)$$

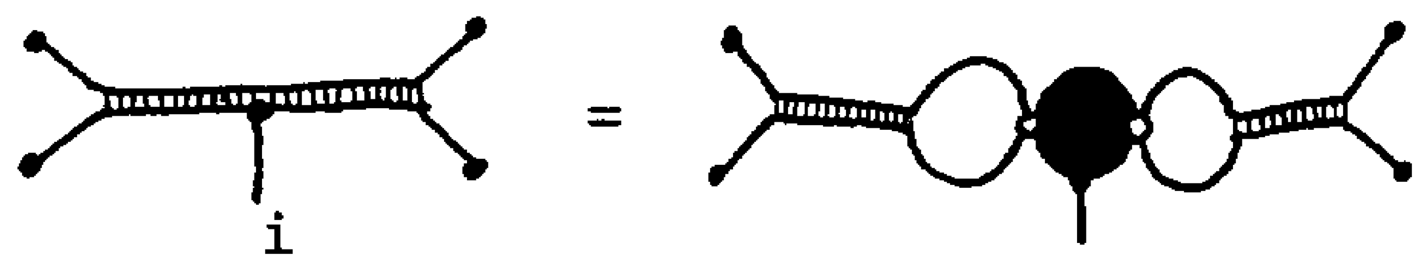
 (A.19)

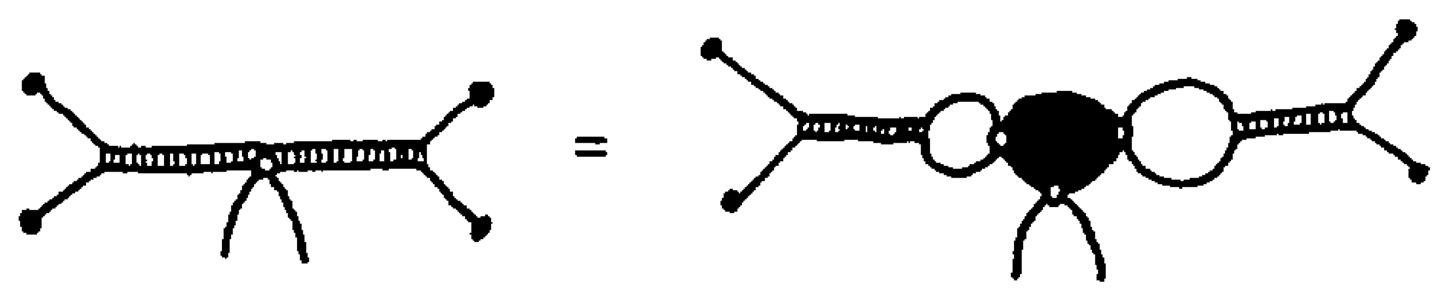
This says that if we follow a line into a connected diagram, we either encounter a 2PI piece, or a 2P-reducible piece.

To be able to evaluate

$$\frac{d}{dJ_i}\frac{d}{dJ_j}\cdot\cdot\frac{d}{dJ_k}W[J]$$

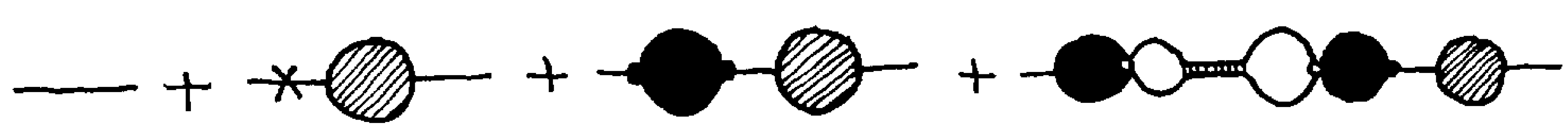
in terms of 2PI bits, we also need to compute $d/d\phi_i\ D_{\underline{k\ell}\ \underline{mn}}$ and $d/dD_{ij}\, D_{\underline{k\ell}\ \underline{mn}}$. They follow from the definition of $D_{\underline{k\ell}\ \underline{mn}}$ as the inverse of $\Gamma_{\underline{k\ell}\ \underline{mn}}$:

$\frac{d}{d\phi_i} D_{\underline{k\ell}\ \underline{mn}} =$ 

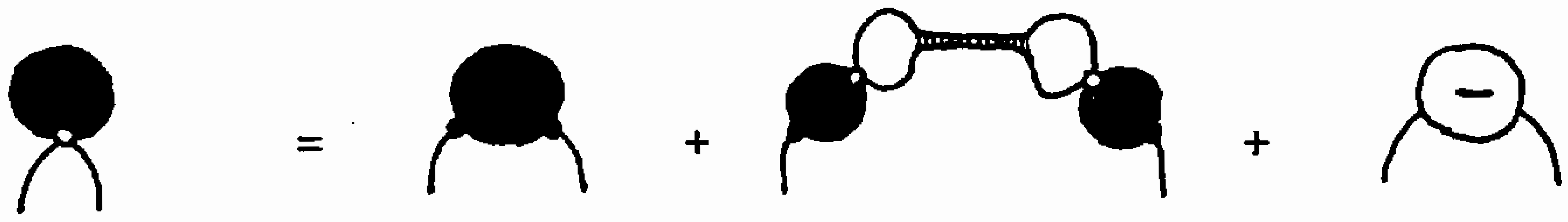
$\frac{d}{dD_{ij}} D_{\underline{k\ell}\ \underline{mn}} =$  (A.20)

This is also sensible, as will be clear from the perturbative expansion of the 2-particle propagator.

Finally, we need to relate d/dD_{ij} (a diagrammatically obscure thing) to $d/d\phi_i\ d/d\phi_j$ (an operation which yields 2PI Green functions). This we obtain by differentiating the first duality relation with respect to d/dJ and using the chain rule

$0 =$  (A.21)

Replacing J_{ij} by the second duality rule and multiplying by inverse propagator, we obtain



$$\frac{d\Gamma[\phi]}{dD_{ij}} = \frac{d^2\Gamma[\phi]}{d\phi_i d\phi_j} + \frac{d^2\Gamma[\phi]}{d\phi_i dD_{k\ell}} D[\phi]_{\underline{k\ell}\,\underline{mn}} \frac{d^2\Gamma[\phi]}{dD_{mn} d\phi_j} + D^{-1}_{ij}[\phi] \;. \qquad \text{(A.22)}$$

This enables us to systematically get rid of d/dD_{ij} derivatives.

Now we can rewrite any relation between connected Green functions in terms of 1PI functions by going from

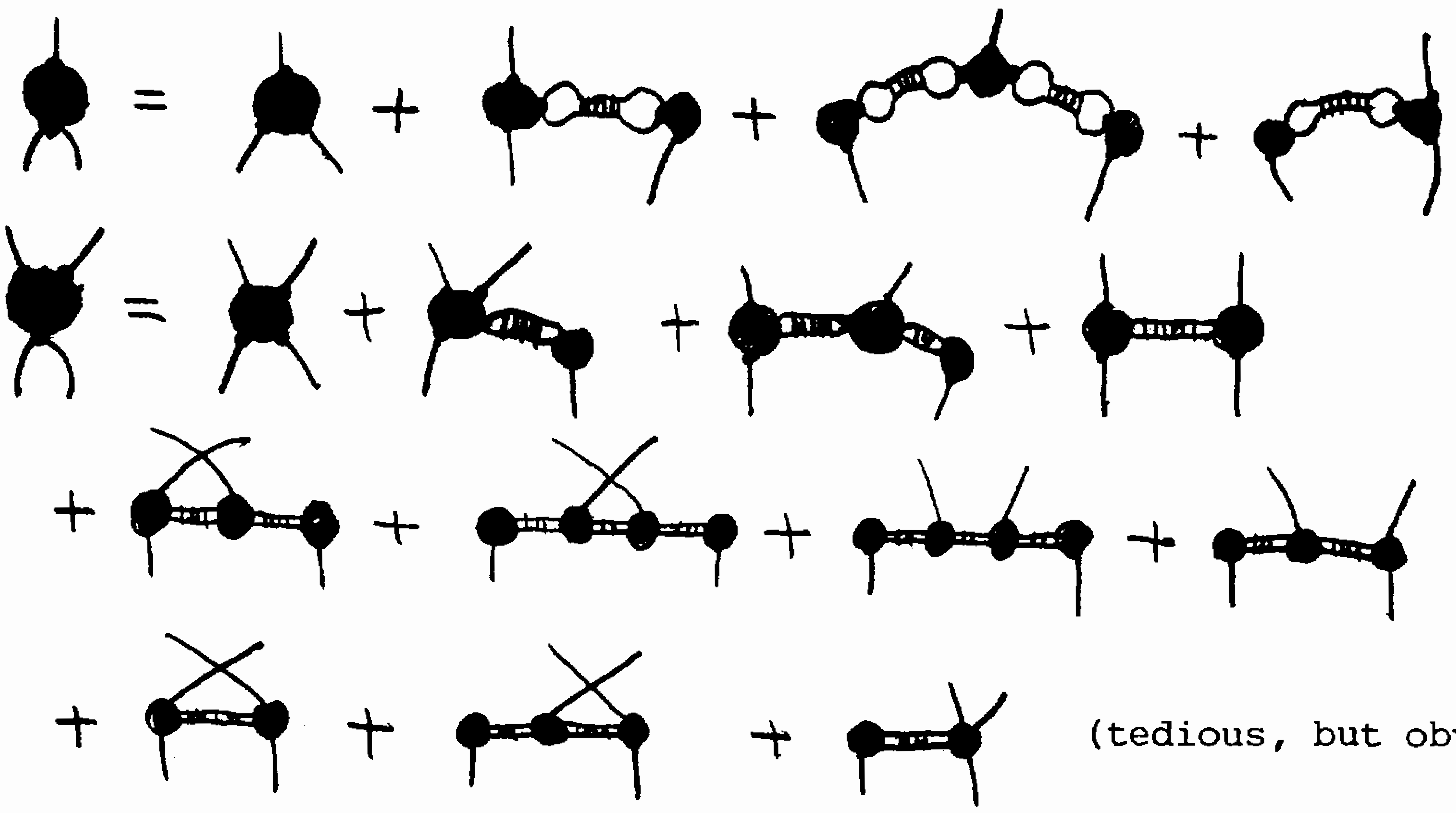
$$J_i, J_{ij}, \frac{d}{dJ_i}, W[J]$$

to dual variables and functions

$$\phi_i, D_{ij}, \frac{d}{d\phi_i}, \Gamma[\phi]$$

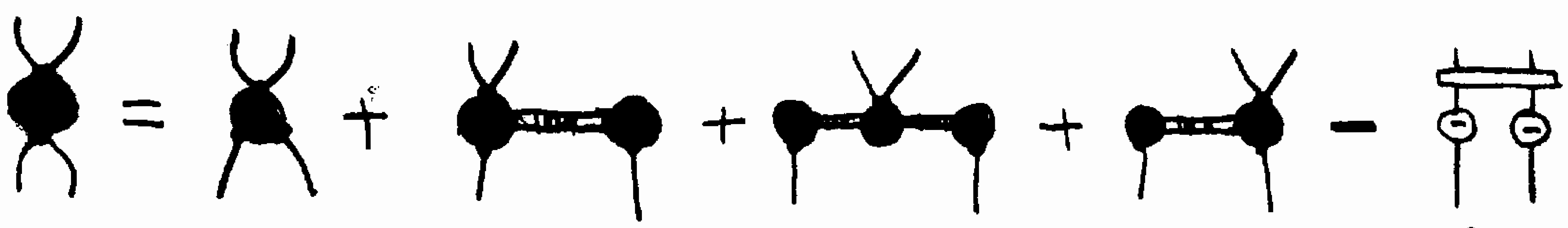
using $D_{\underline{ij}\,\underline{k\ell}}$ and d/dD_{ij} in intermediate steps.

Sundry expansions:



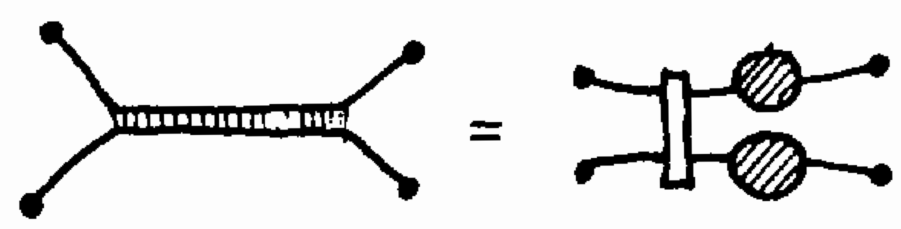
(A.23)

(tedious, but obvious)



(A.24)

The last term shows that not only is $\Gamma_{\underline{ij}\,\underline{k\ell}}$ not 2P-irreducible, it is not even connected. That is a good thing; it is necessary so that $D_{\underline{ij}\,\underline{k\ell}}$ can be the inverse of $\Gamma_{\underline{ij}\,\underline{k\ell}}$: it has to start as

 + (connected pieces) (A.25)

Perturbative expansions for 2PI graphs

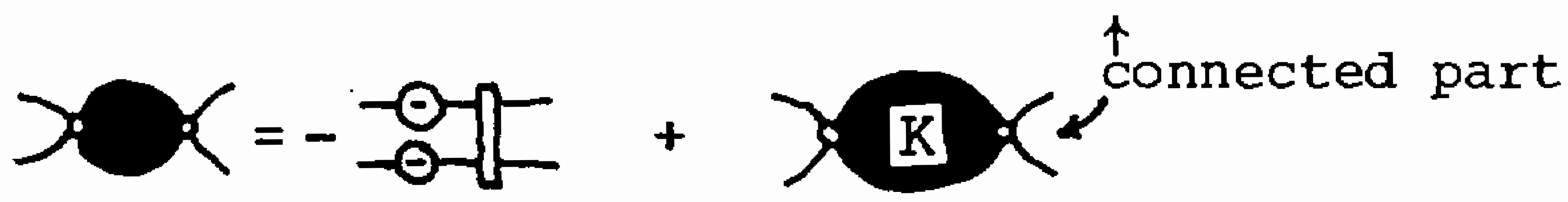
Perturbative expansions isolate the quadratic part of the action (bare propagator) and treat the rest as "interaction" parts. In the general formalism, the bare propagator is hidden in

$$\Gamma_{ij} = -\Delta^{-1}_{ij} + \pi_{ij}$$

(A.26)

It is convenient to also isolate the non-interacting part of the two-particle propagator:

$$\Gamma_{\underline{ij}\ \underline{k\ell}} = -\frac{1}{2}\left(D^{-1}_{ik}D^{-1}_{j\ell} + D^{-1}_{i\ell}D^{-1}_{jk}\right) + K_{\underline{ij}\ \underline{k\ell}}$$

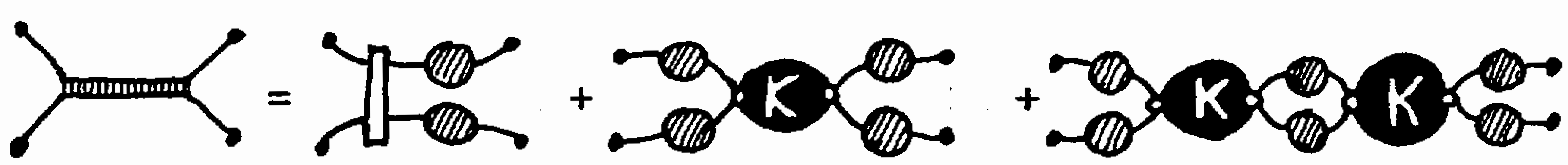


(A.27)

We implement these reshufflings by defining an "interaction" general functional

$$\Gamma[\phi] = \Gamma^{(I)}[\phi] - \frac{1}{2}\phi_i \Delta^{-1}_{ij}\phi_j + \frac{1}{2}\,\mathrm{tr}\,\ell n D \ . \qquad \text{(A.28)}$$

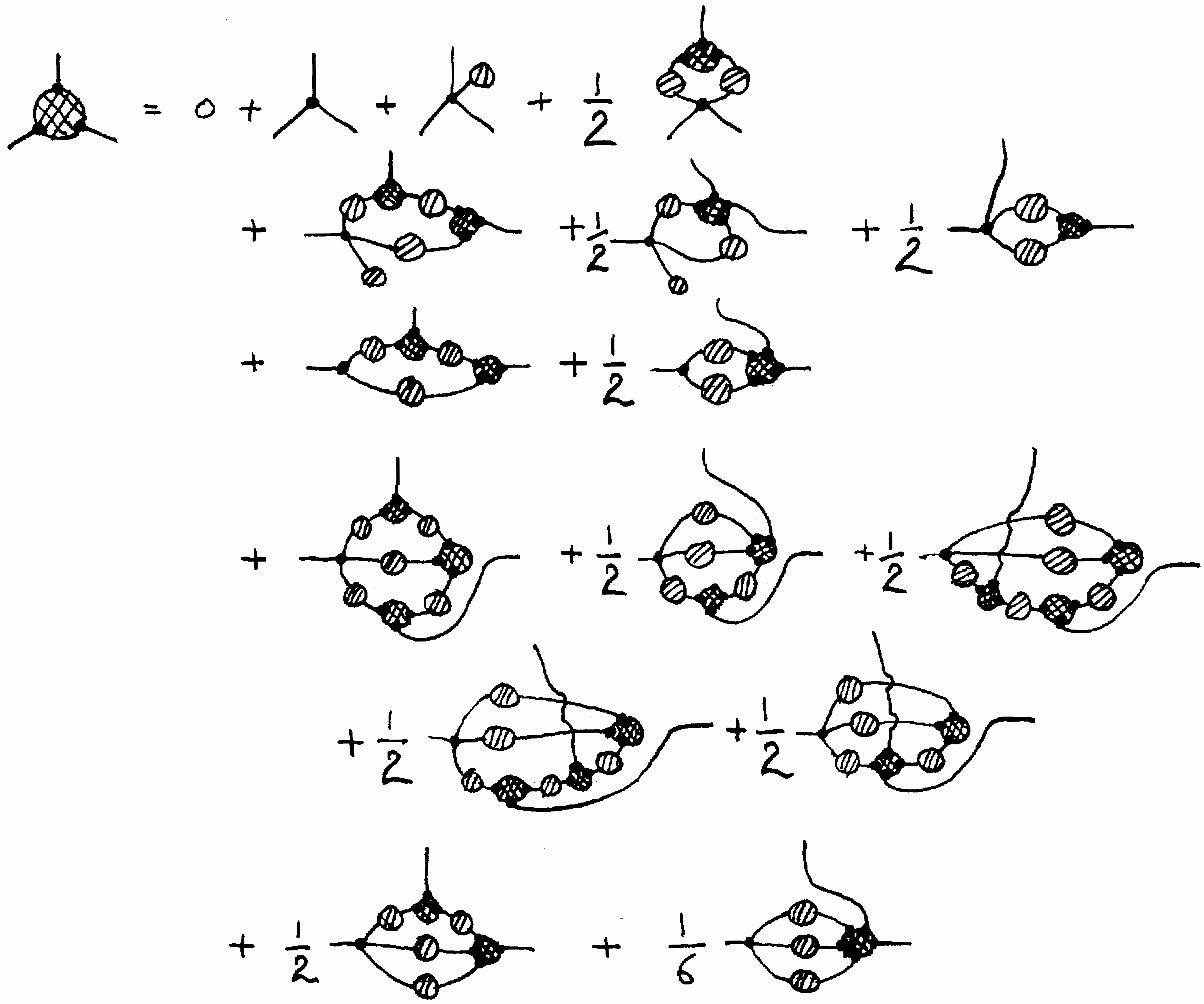
Now we can expand the two-particle propagator in terms of $K_{\underline{ij}\ \underline{k\ell}}$:



(A.29)

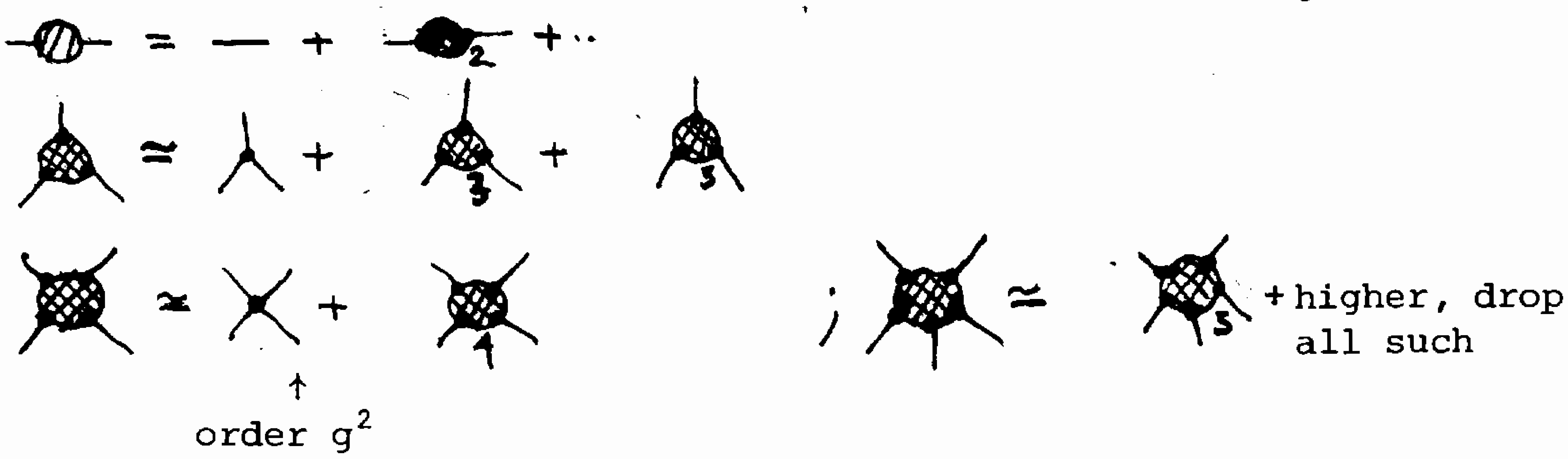
APPENDIX B: Solution of "find 7 errors" (Exercise 2.H.3)

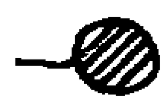
Pull a third leg out of the equation (2.35):

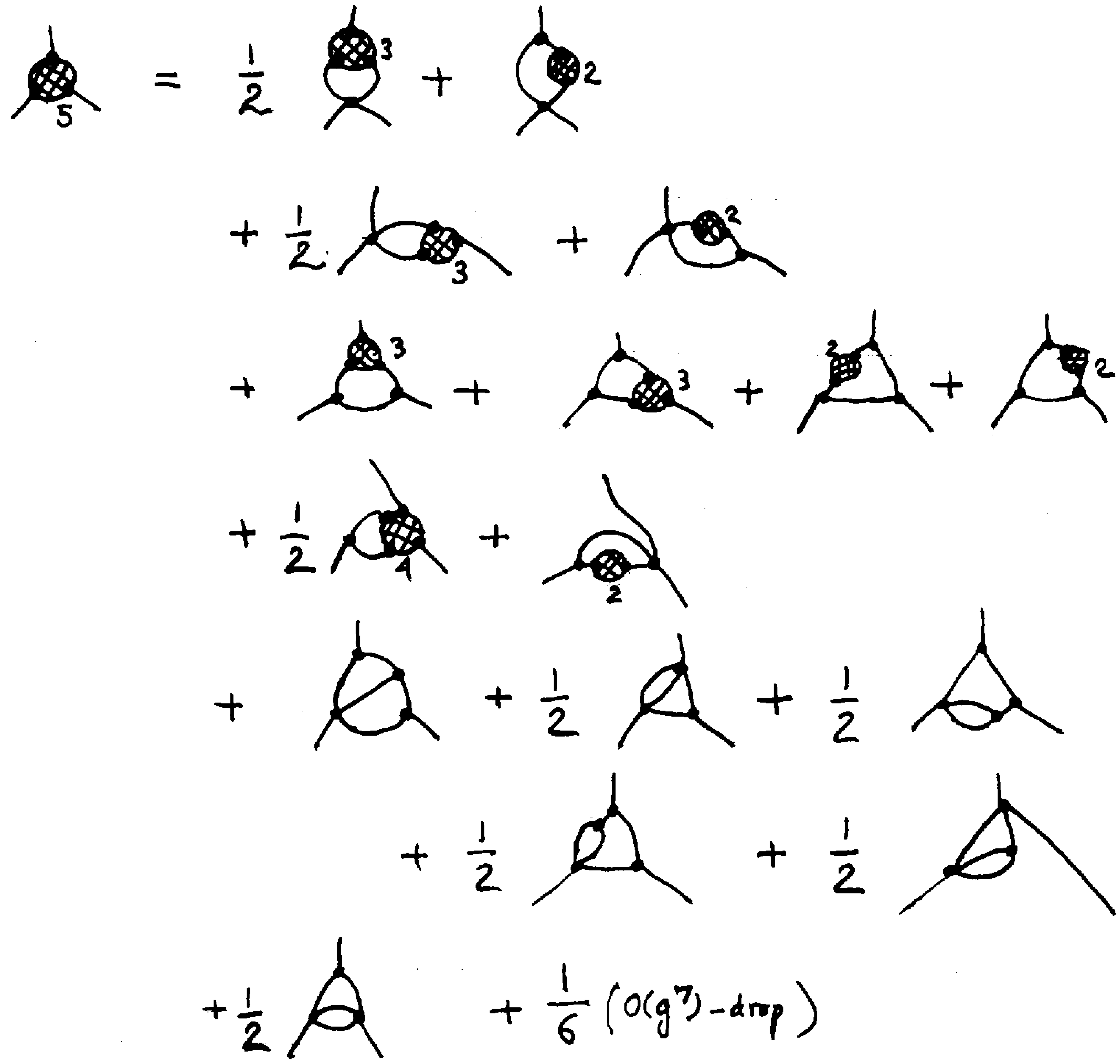


As we need Γ_{ijk} only to g^5 order, truncate all subdiagram expansions

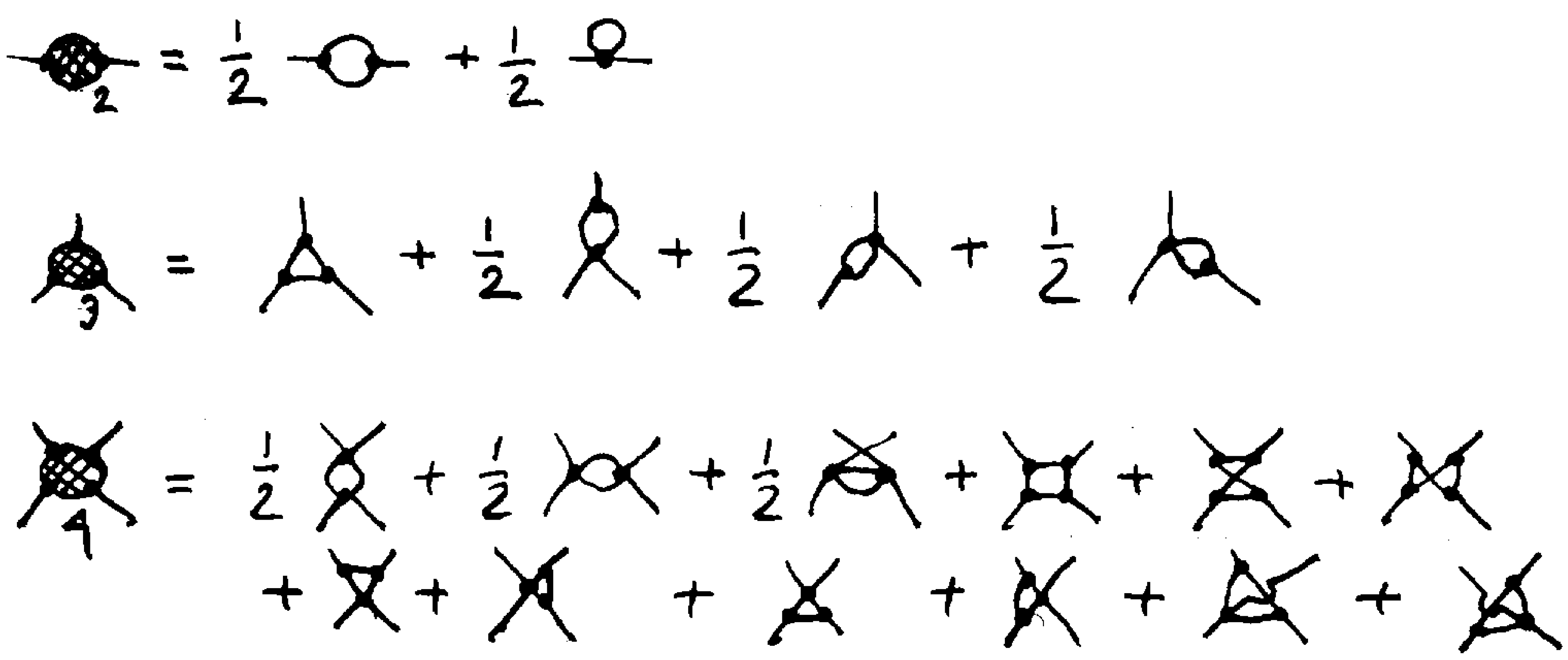
(where subscript k means all terms of order g^k)

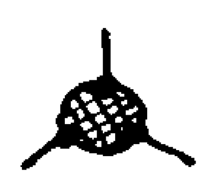
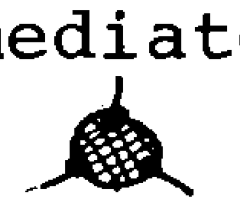


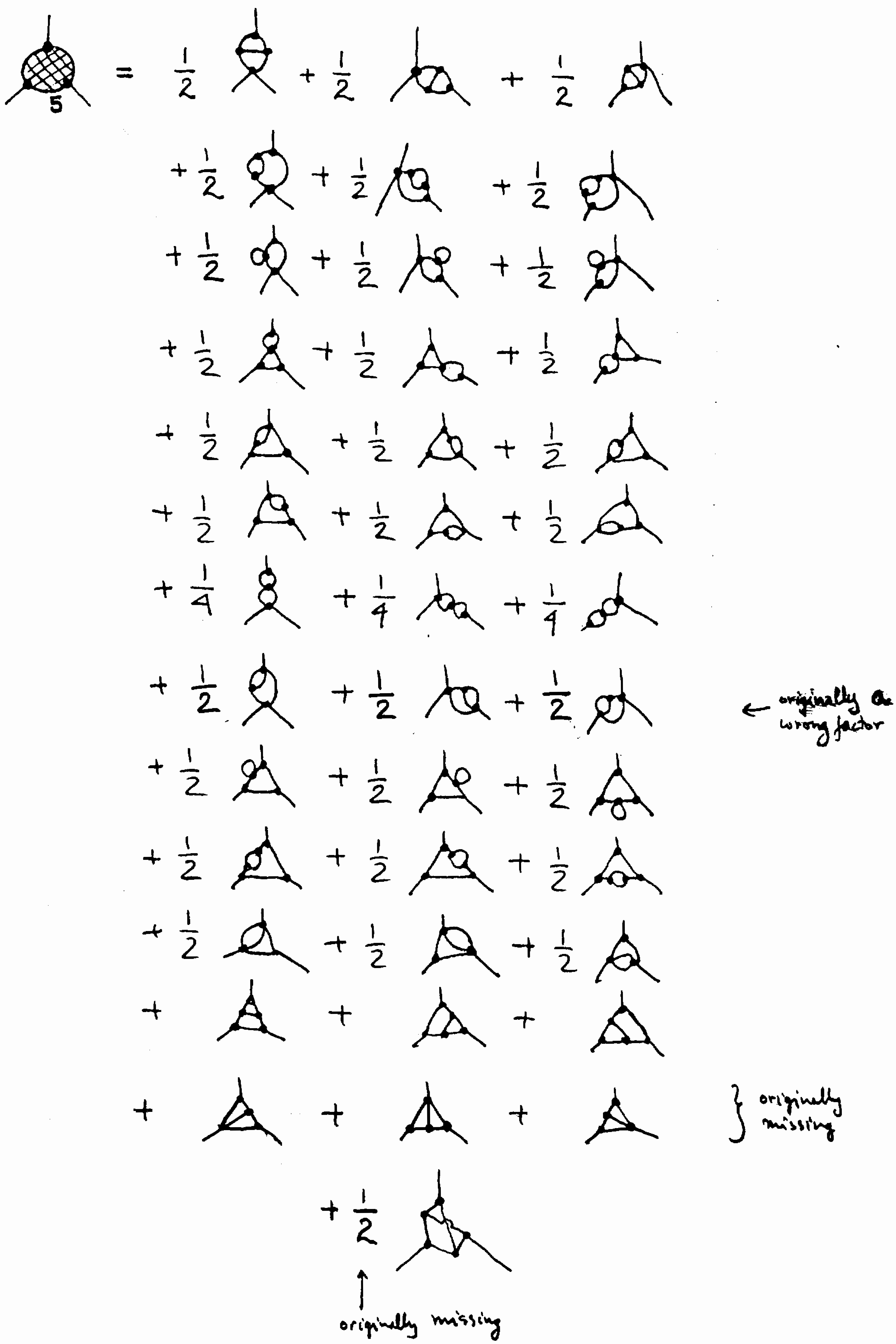
Substituting such expansions, and keeping only g^5 terms: (remember,  = 0)



Need subdiagram expansions



The last expansion comes from pulling out a leg from  and immediately dropping all terms higher than g^1 (the last 7 terms in  Dyson-Schwinger equation). Substituting, one obtains the correct expansion:


5
originally the wrong factor
originally missing
originally missing

APPENDIX C: SOME POPULAR GAUGES

Covariant gauges: (Feynman a = 1; Landau a = 0):

$$D^{\mu\nu} = \frac{-i}{k^2+i\varepsilon}\left[g^{\mu\nu} - (1-a)\frac{k^\mu k^\nu}{k^2}\right] , \quad h^\mu = k^\mu \quad . \tag{C.1}$$

General axial gauges:

$$D^{\mu\nu} = -\frac{i}{k^2+i\varepsilon}\left[g^{\mu\nu} - \frac{n^\mu k^\nu + k^\mu n^\nu}{n\cdot k} + \frac{ak^2+n^2}{(n\cdot k)^2} k^\mu k^\nu\right] , \quad h^\mu = \frac{k^2 n^\mu}{(k\cdot n)} . \tag{C.2}$$

Usually $n_\mu = (0,0,0,1)$ picks out a spatial axis.

Axial or temporal gauges (a=0):

$$D_{\mu\nu} = \frac{-i}{k^2+i\varepsilon}\left[g_{\mu\nu} - \frac{n_\mu k_\nu + k_\mu n_\nu}{(n\cdot k)} + \frac{n^2 k_\mu k_\nu}{(n\cdot k)^2}\right] , \quad n_\mu D^{\mu\nu} = 0 . \tag{C.3}$$

General planar gauges:

$$D_{\mu\nu} = \frac{-i}{k^2+i\varepsilon}\left[g_{\mu\nu} - \frac{n_\mu k_\nu + k_\mu n_\nu}{(n\cdot k)} + (1-a)\frac{n^2 k_\mu k_\nu}{(n\cdot k)^2}\right] , \quad h^\mu = \frac{k^2 n^\mu}{(k\cdot n)} \quad . \tag{C.4}$$

Lightcone ($a=0$, $n^2=0$) and planar ($n^2 = -ak^2 \neq 0$) gauges:

$$D_{\mu\nu} = \frac{-i}{k^2+i\varepsilon}\left[g_{\mu\nu} - \frac{n_\mu k_\nu + k_\mu n_\nu}{(n\cdot k)}\right] \quad . \tag{C.5}$$

General Coulomb gauges:

$$D^{\mu\nu} = \frac{-i}{k^2+i\varepsilon}\left[g^{\mu\nu} - \frac{n\cdot k(n^\mu k^\nu + k^\mu n^\nu)}{(n\cdot k)^2 - n^2k^2} - \frac{ak^2 - n^2((n\cdot k)^2 - n^2k^2)}{((n\cdot k)^2 - n^2k^2)^2} k^\mu k^\nu\right] ,$$

$$h^\mu = k^2\frac{(n\cdot k)n^\mu - k^\mu}{(n\cdot k)^2 - k^2} , \quad h^\mu n_\mu = 0 . \tag{C.6}$$

Usually $n^\mu = (1,0,0,0)$ picks out the time direction so that $k^\mu - (n\cdot k)n^\mu = (0,\vec{k})$.

Coulomb gauge $(a=0\ ,\ n^2=1)$:

$$D_{\mu\nu} = -\frac{i}{k^2+i\varepsilon}\left[g_{\mu\nu} - \frac{(n\cdot k)(k_\mu n_\nu + n_\mu k_\nu) - k_\mu k_\nu}{(n\cdot k)^2 - k^2}\right] \quad . \tag{C.7}$$

See (6.63) for the gauge-fixing terms $\mathcal{L}_{fix}$.

APPENDIX D: FEYNMAN RULES FOR QCD

. <u>Propagators</u>

gluons  $= \delta_{ij} \frac{-i}{k^2} \left(g^{\mu\nu} + f^\mu f^\nu + k^\mu f^\nu\right)$, (D.1a)

 $= \delta_{ij} \frac{-i}{k^2} g^{\mu\nu}$ Feynman gauge . (D.1b)

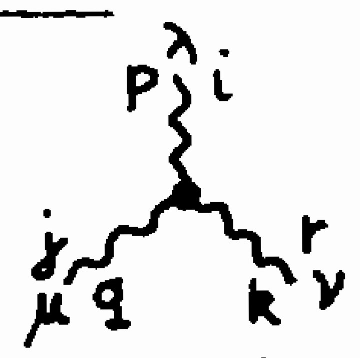
ghost  $= \delta_{ij} \frac{-i}{k^2}$ (D.2)

quark 

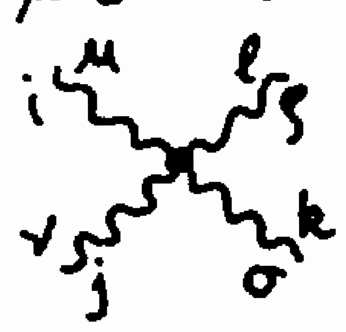
$= \delta^a_b \frac{i}{\not{p}-m}$ (D.3)

(cf. appendix C for other gauges).

<u>Vertices</u>



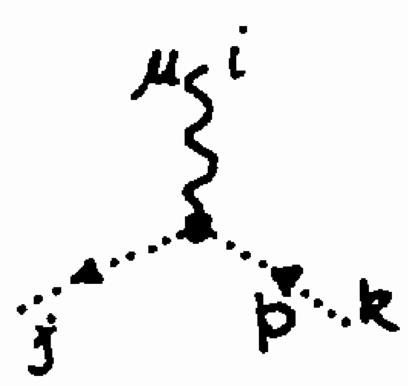
$= (-igC_{ijk})\, i[g_{\lambda\mu}(p-q)_\nu g_{\mu\nu}(q-r)_\lambda + g_{\nu\lambda}(r-p)_\mu]$, (D.4)



$= (-g^2 C_{ijm} C_{mk\ell})\, i(g_{\mu\sigma} g_{\nu\rho} - g_{\mu\rho} g_{\nu\sigma})$

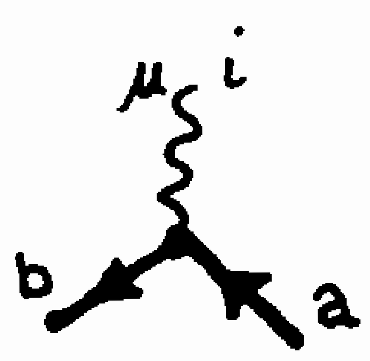
$+ (-g^2 C_{\ell im} C_{mjk})\, i(g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma})$

$+ (-g^2 C_{ikm} C_{mj\ell})\, i(g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma})$. (D.5)



$= (-igC_{ijk})(-ih^\mu)$ (D.6a)

$= (-igC_{ijk})(-ip^\mu)$ (covariant gauges) (D.6b)



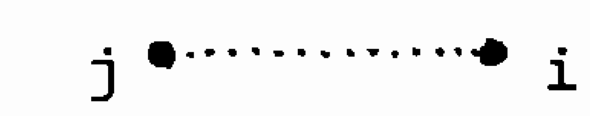
$= g\left(T_i\right)^a_b (-i\gamma^\mu)$. (D.7)

All momenta flow outward. Ghost vertices for other gauges are given in appendix C. The first factor is the color weight (cf. exercises 6.E.1 and 6.F.4). A factor $\int \frac{d^d k}{(2\pi)^d}$ for each loop.

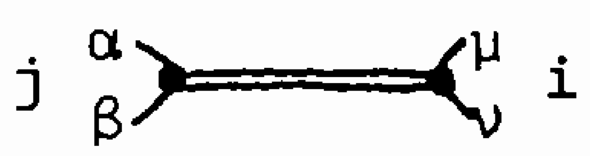
Diagrammatic notation

One way to avoid the proliferation of color indices, Minkowski indices, and other QCD factors is to introduce auxiliary Feynman rules:

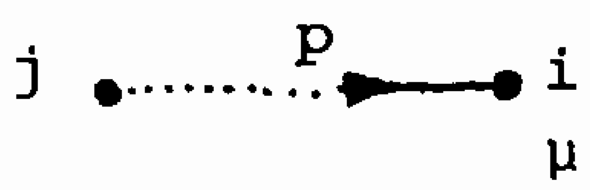
Auxiliary propagators:



$$\delta_{ij}\,\frac{-i}{p^2} \qquad \text{(D.8a)}$$

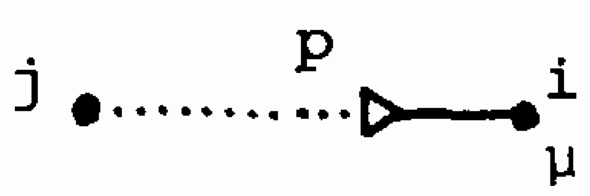


$$\delta_{ij}\,\frac{-i}{p^2}\,g_{\alpha\mu}g_{\beta\nu} \qquad \text{(D.8b)}$$



$$\delta_{ij}\,\frac{-i}{p^2}\,(\pm p_\mu) \qquad \text{(D.9a)}$$

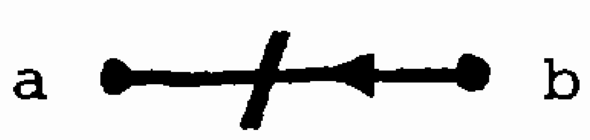
+ if arrow along p
- if arrow against p



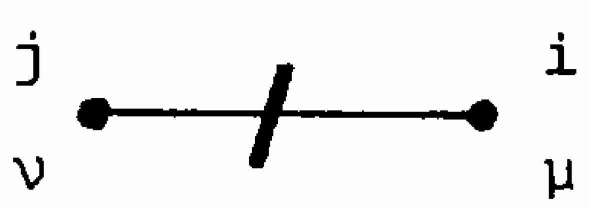
$$\delta_{ij}\,\frac{-i}{p^2}\,(\pm h_\mu(p)), \qquad \text{(D.9b)}$$

+ if arrow along p
- if arrow against p

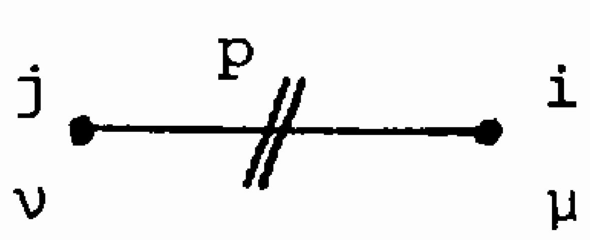
$$\delta_{ij}\,\frac{-i}{p^2}\,g_{\nu\mu}(\pm p_\mu) \qquad \text{(D.9c)}$$



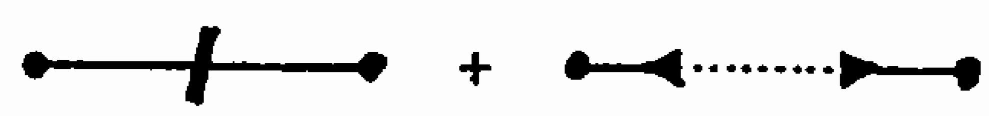
$$\delta^b_a\; i \qquad \text{(D.10a)}$$



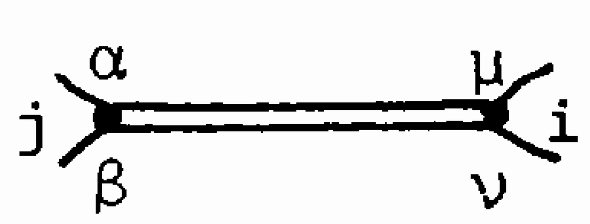
$$\delta_{ij}(-i)g_{\mu\nu} \qquad \text{(D.10b)}$$



$$\delta_{ij}(-i)(g_{\mu\nu} - p_\mu p_\nu/p^2)$$

=  (D.10c)

$$\delta_{ij}(-i) \qquad \text{(D.10d)}$$

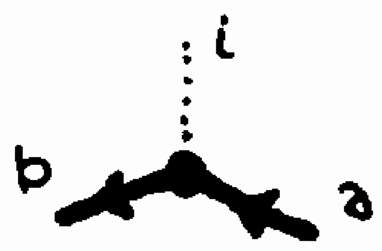


$$\delta_{ij}(-i)g_{\alpha\mu}g_{\beta\nu} \qquad \text{(D.10e)}$$

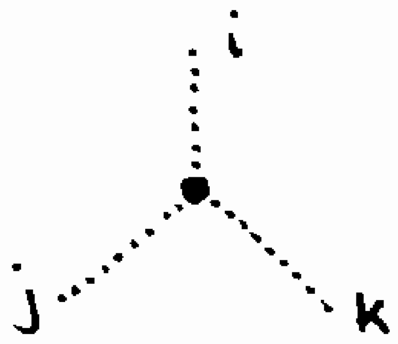
 = −  (D.10f)

Each line connecting two vertices (or an external source and a vertex) carries factor $-i/p^2$ for gluons and ghosts, and $i/(\not{p}-m)$ for quarks. Dotted lines keep track of color indices; thin lines keep track of Minkowski indices.

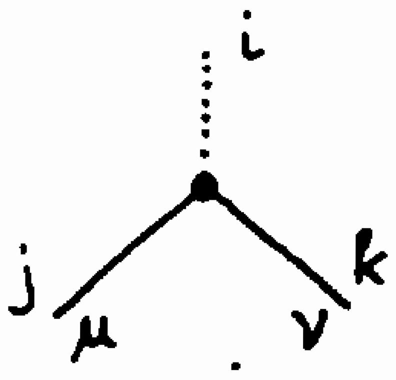
Auxiliary vertices:



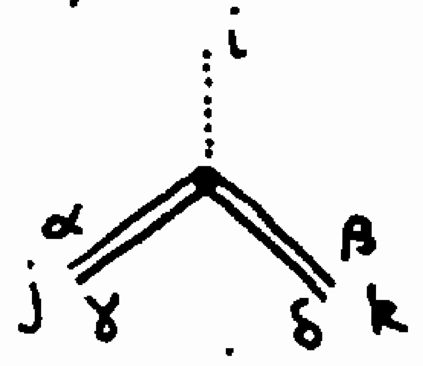
$\left(T_i\right)^a_b\,(-i)$ (D.11a)



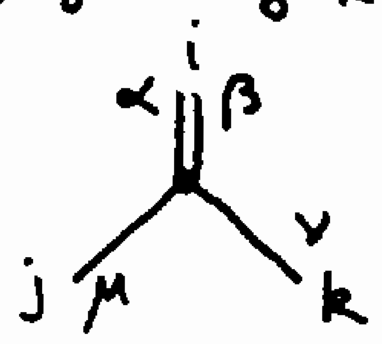
$(-iC_{ijk})i$ (D.11b)



$(-iC_{ijk})ig_{\mu\nu}$ (D.11c)



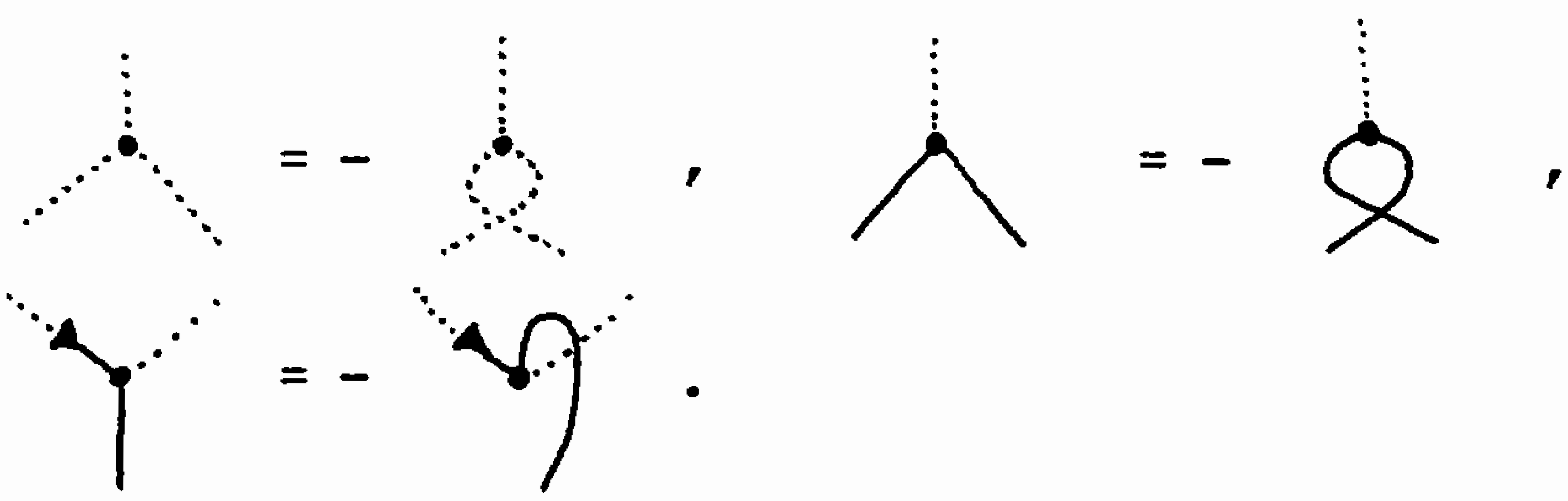
$(-iC_{ijk})ig_{\alpha\beta}g_{\gamma\delta}$ (D.11d)



$(-iC_{ijk})ig_{\alpha\mu}g_{\beta\nu}$ (D.11e)

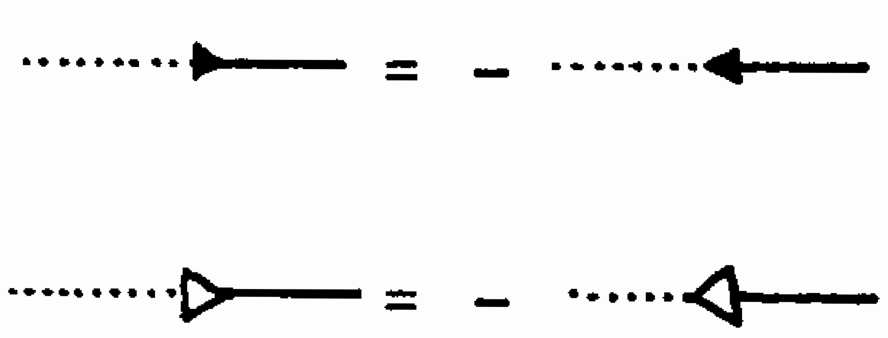
Signs

C_{ijk} indices are read anticlockwise around the vertex. Due to the antisymmetry of C_{ijk}, the corresponding vertices change sign under interchange of any two legs:



(D.12)

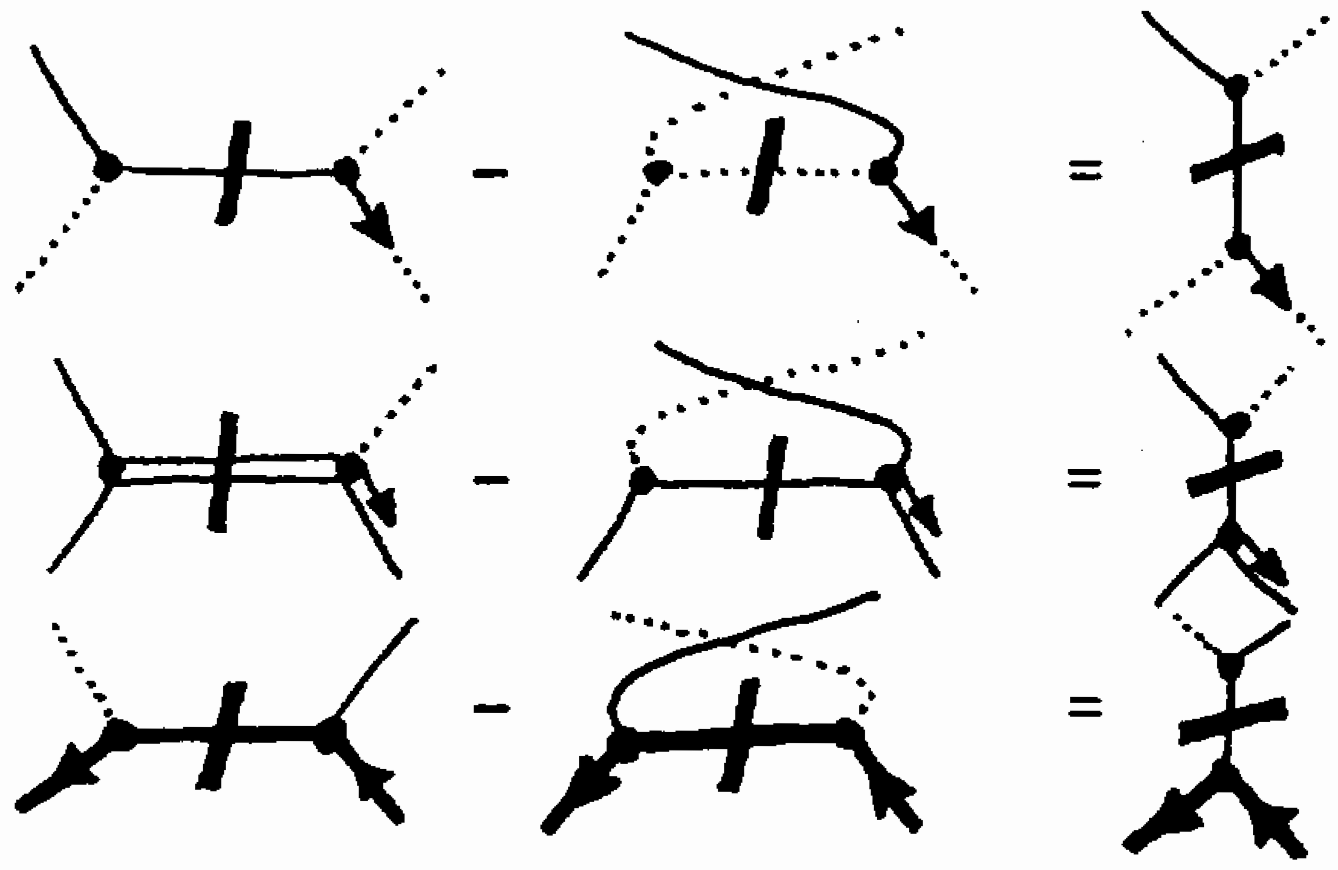
Arrows for p^μ and h^μ factors indicate the momentum flow and change sign under arrow reversal:



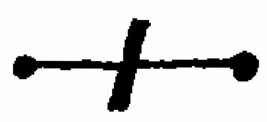
(D.13)

Jacobi identities, Lie algebra

They are all statements of (6.48) and (6.21), but decorated with different Minkowski factors:



(D.14)

Comments: It would be more consistent to treat propagators as two-leg vertices, but it is traditional to denote them by lines. This causes some unnecessary ugliness, such as slash notation  for lines without propagators, and confusion between  and  which we tried to clarify in equation (7.20).