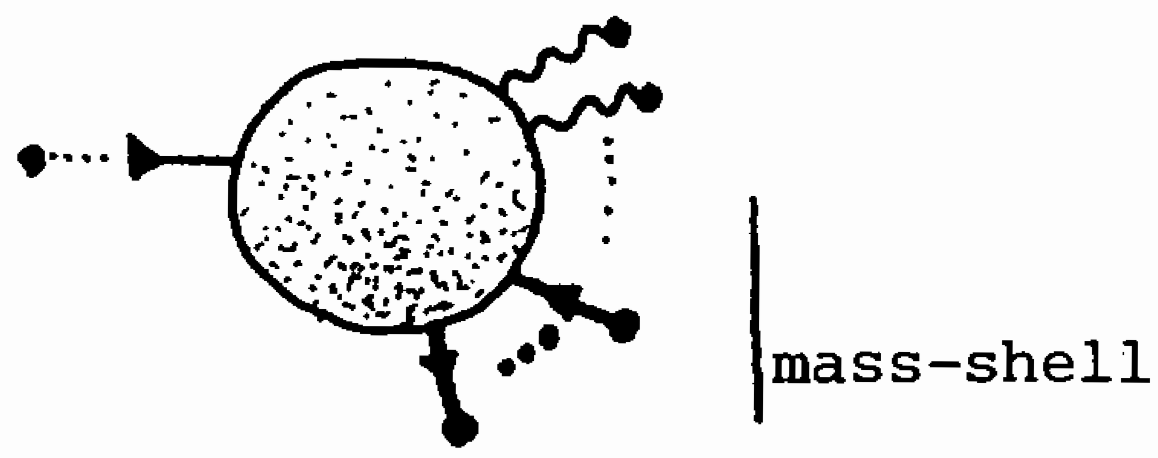
7. QCD WARD IDENTITIES

We tried to put a little color into QED and we got into a considerable mess. It seems as though one has to introduce a new vertex or particle for each process one looks at - a dismal prospect. Fortunately things are not that bad - we shall now prove that with the QCD vertices constructed in the last chapter the gaugeons decouple from all S-matrix elements. Regardless of their later guises, the requisite identities are contained in the original Gerard 't Hooft's paper[†], so we shall call them Ward identities.

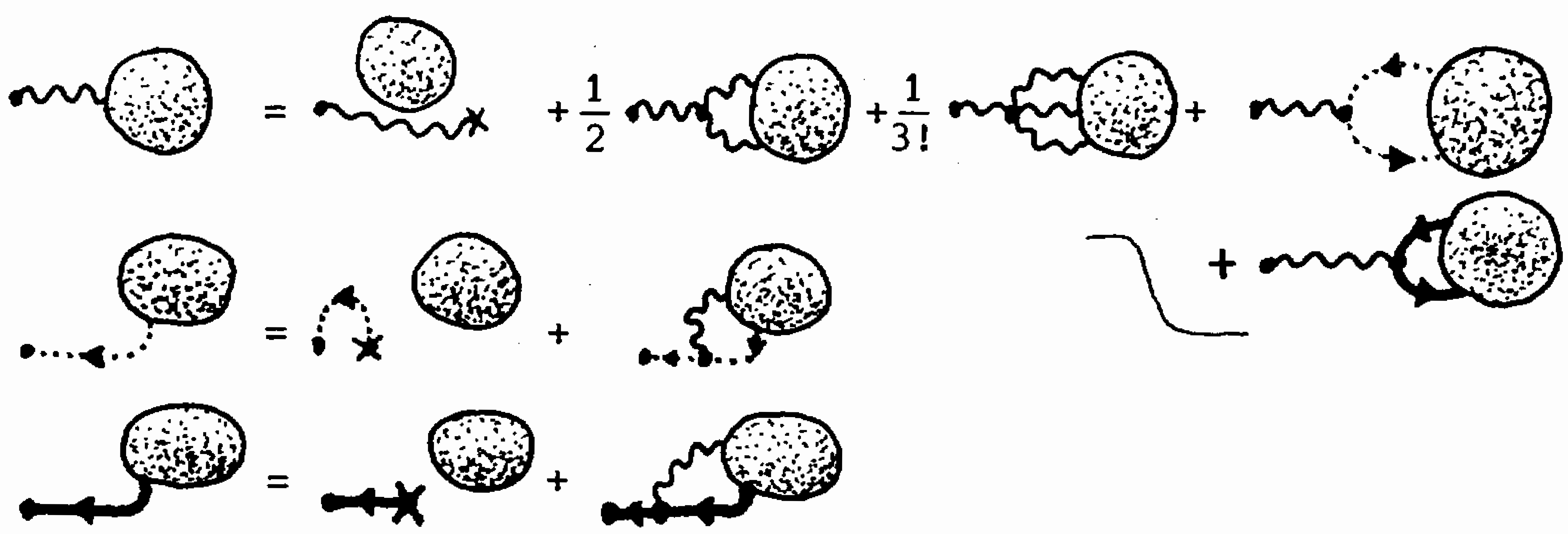
A. Ward identities for full Green functions

In this section we shall prove that the gaugeons (6.5) decouple from any QCD mass-shell process;



= 0 (7.1)

The QCD Dyson-Schwinger equations (2.12)



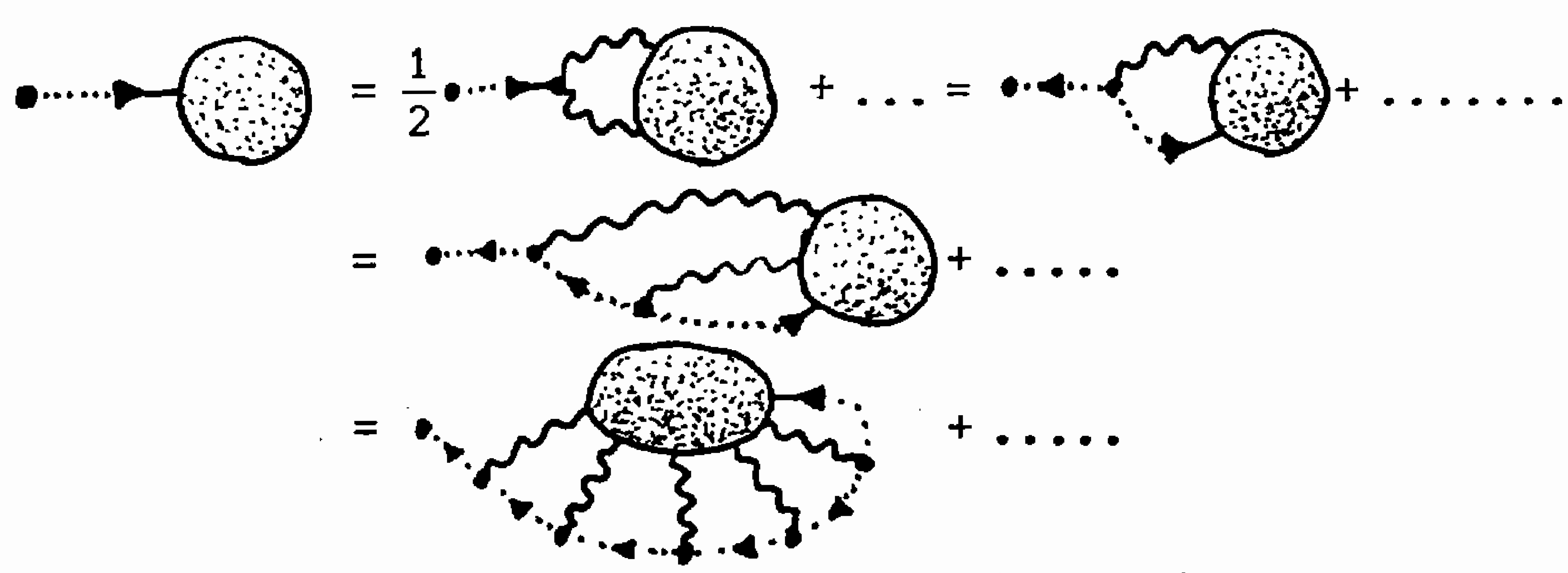
(7.2a)

(7.2b)

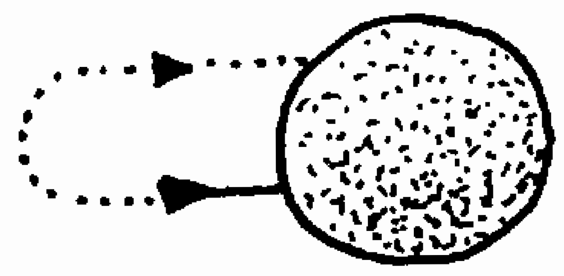
(7.2c)

enable us to follow the gaugeon into the Green functions. Because of the bare 3-gluon Ward identity (6.37), the gaugeons "propagate" into the diagrams:

[†] G. 't Hooft, "Renormalization of massless Yang-Mills fields", Nucl. Phys. B33(1971)173. These identities are also known as Lie - Engel - Schur - Wigner - Eckhart - Schwinger - Stückelberg - Feynman - Ward - Takahashi - Green - 't Hooft - Veltman - Taylor - Slavnov - Lee - Zinn-Justin - Nielsen - Kluberg - Stern - Zuber - Becchi - Rouet - Stora - Kugo-Ojima - Feigenbaum - Witten - Polyakov - Parisi - Wilson - Moffat identities.

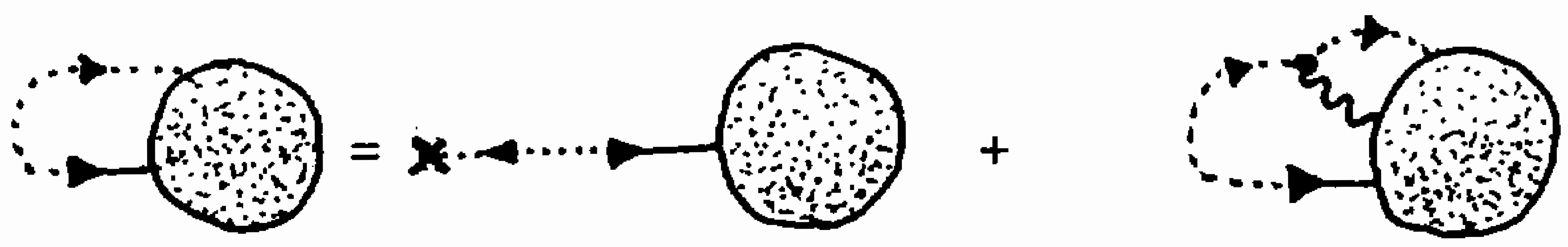


This fact (together with much hindsight) suggests that the convenient starting point for the proof is not the external leg gaugeon (7.1), but gaugeon insertion anywhere inside a Green function:



(7.3)

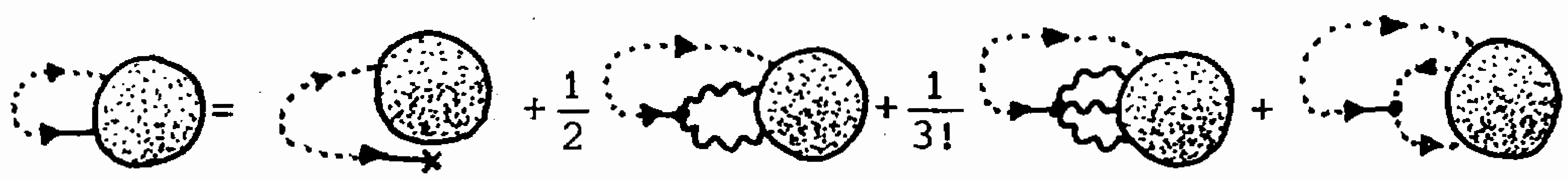
The ghost DS equation (7.2b) yields the desired external gaugeon insertion (7.1), together with an extra term



(7.4)

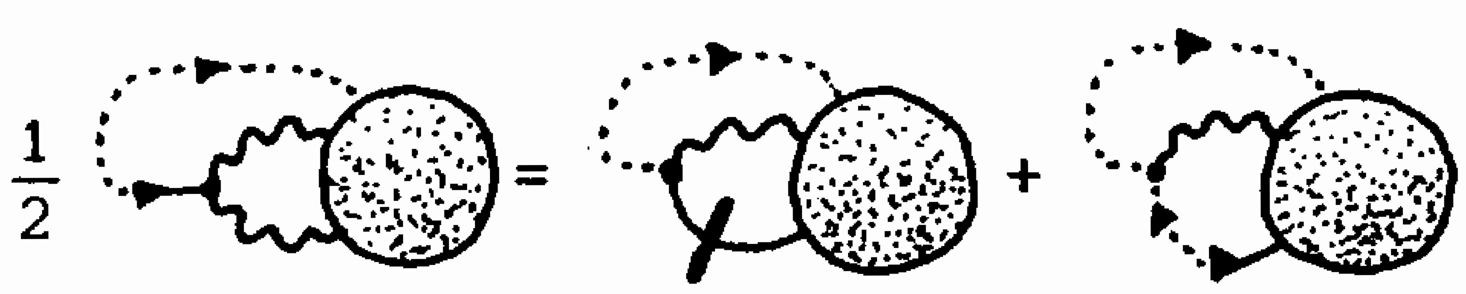
> As ghosts are fermions, the ghost equations are bound to cause sign anxieties. The best thing to do is to relax and remember that the only thing that matters is that each ghost loop carries a minus sign.

The gluon DS equation (7.2a) yields



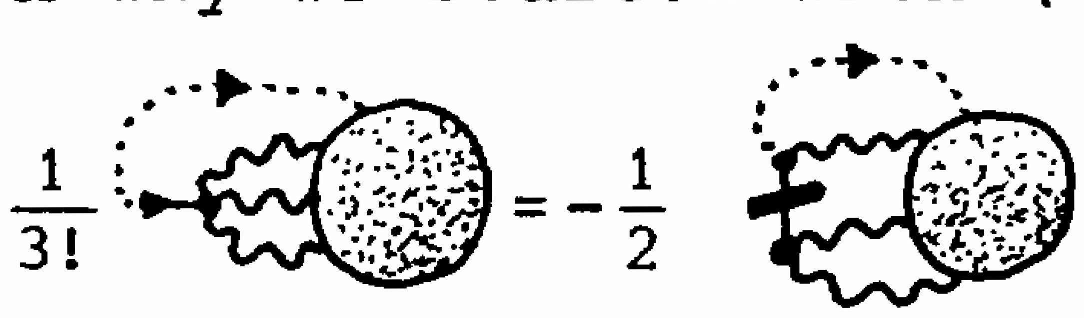
(7.5)

(We omit quarks for the time being - their inclusion is straightforward, cf. exercise 7.A.1). The last three terms are clearly there to be hit by the bare Ward identities (6.37), (6.54), and (6.42):

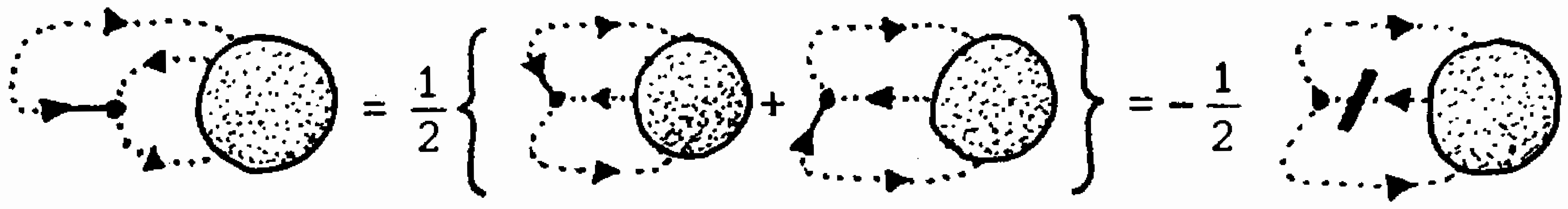


(7.6)

(The second term cancels the extra bit in (7.4); this is the reason why we started with (7.3) rather than (7.1).)

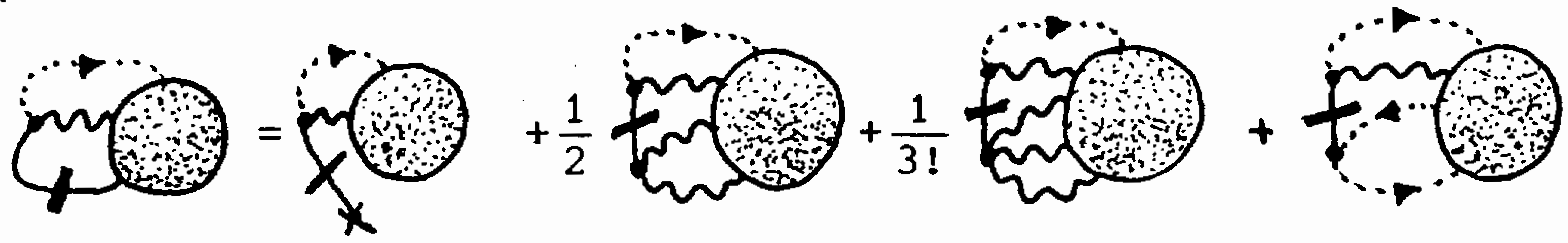


(7.7)



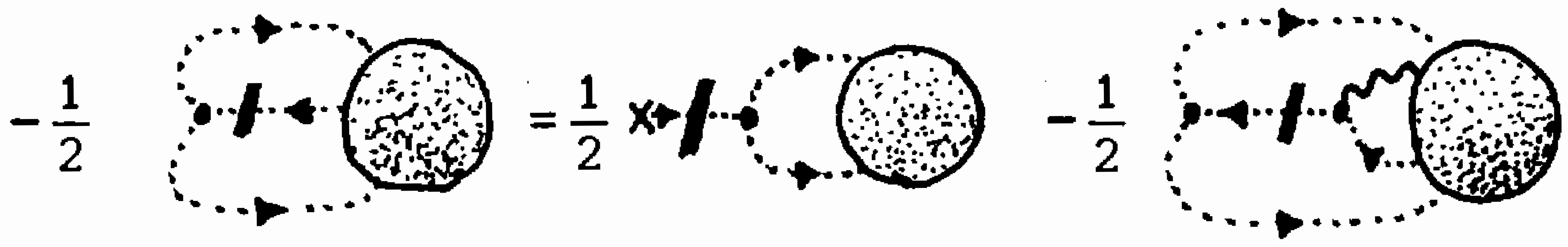
(7.8)

We turn back to DS equations to expand the surviving term in (7.6):



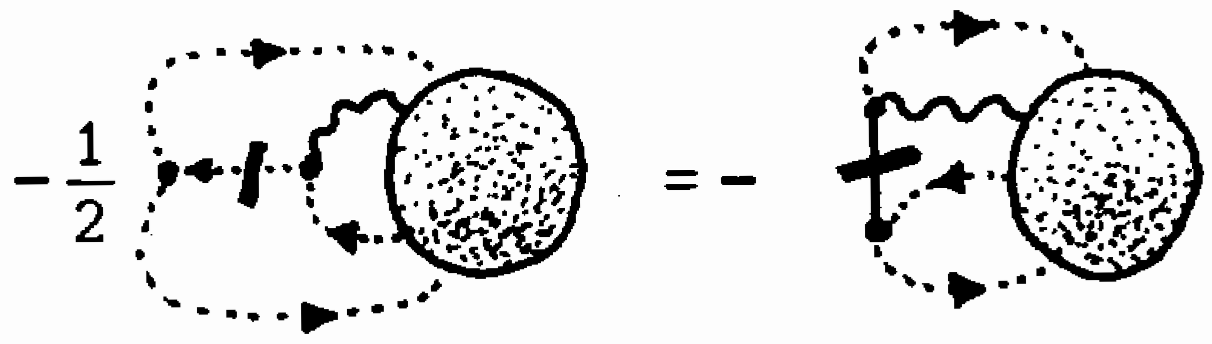
(7.9)

The second term cancels against (7.7), the third term vanishes by (6.55), and to kill the last term we expand (7.8)



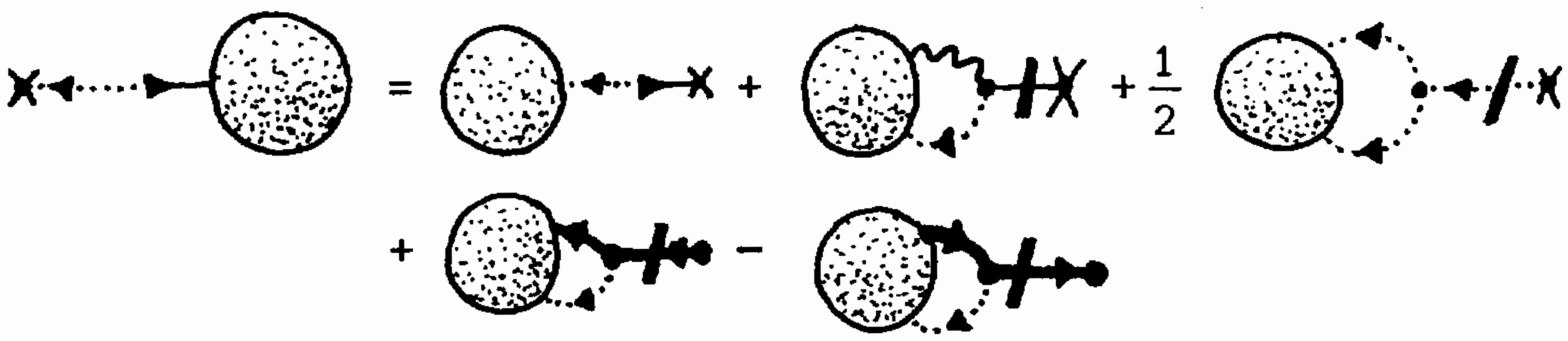
(7.10)

By the Jacobi identity (6.66) the second term cancels the last term in (7.9)



(7.11)

All the messy terms have cancelled. We collect the survivors, putting (7.4) on the left-hand side and (7.5) on the right-hand side:



(7.12)

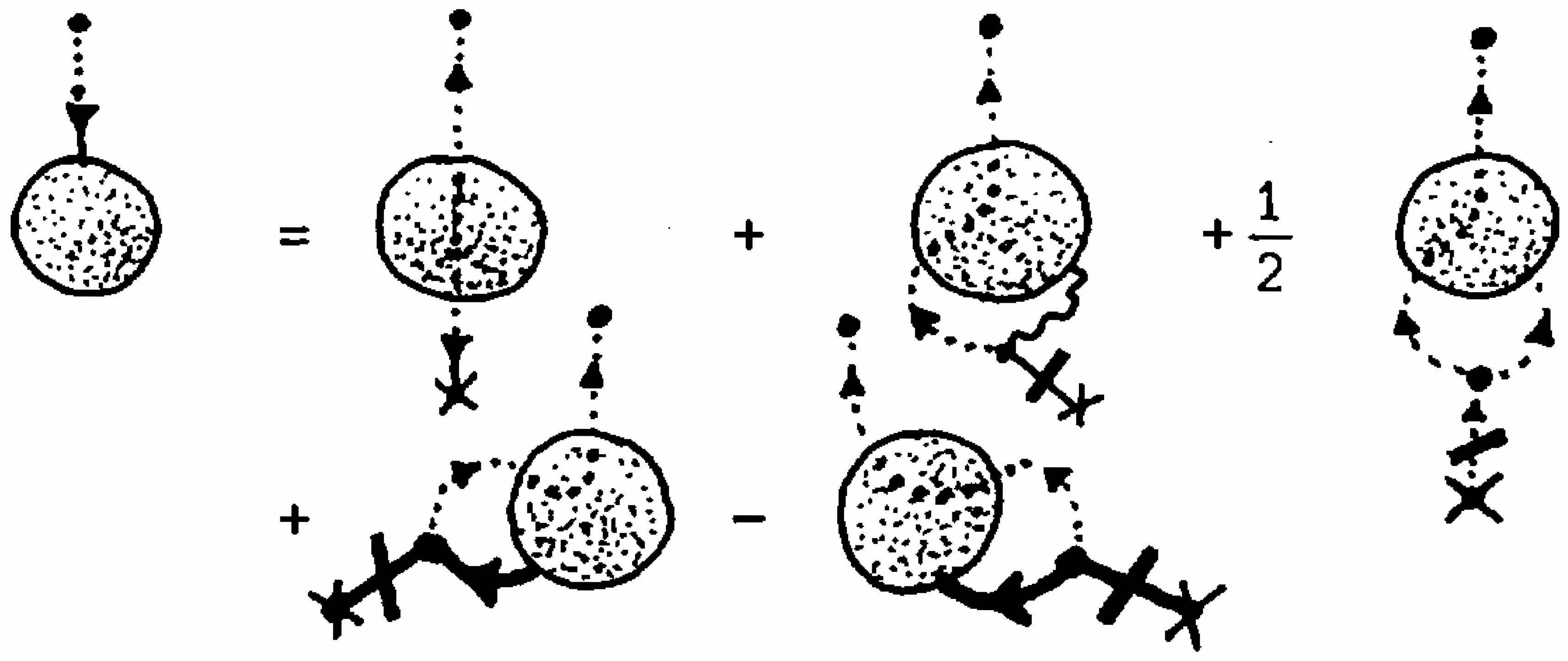
(we have included the quarks - cf. exercise 7.A.1). This is our main result; the <u>Ward identities</u> for the full Green functions. In (7.1) we set out to prove that the left-hand side (a gaugeon insertion) vanishes for any mass-shell process. All the terms on the right-hand side vanish on the mass-shell; the first by the polarization condition (6.1) and the remainder by the equations of motion (6.2), so the gaugeons indeed decouple.

<u>Exercise 7.A.1</u> <u>Quark Ward identities</u>. Derive (7.12) by keeping the quark terms in DS equations and using the bare quark Ward identity (6.7).

<u>Exercise 7.A.2</u> <u>Inevitability of ghosts</u>. Try to check the gaugeon decoupling in the theory without ghosts (drop (7.2b) and the ghost term in (7.2a)). Do the non-vanishing terms suggest introduction of ghosts?

B. Examples of Ward identities

The Ward identities (7.12) can be rewritten in a more transparent form by pulling out an anti-ghost leg and setting the remaining anti-ghost sources equal to zero:

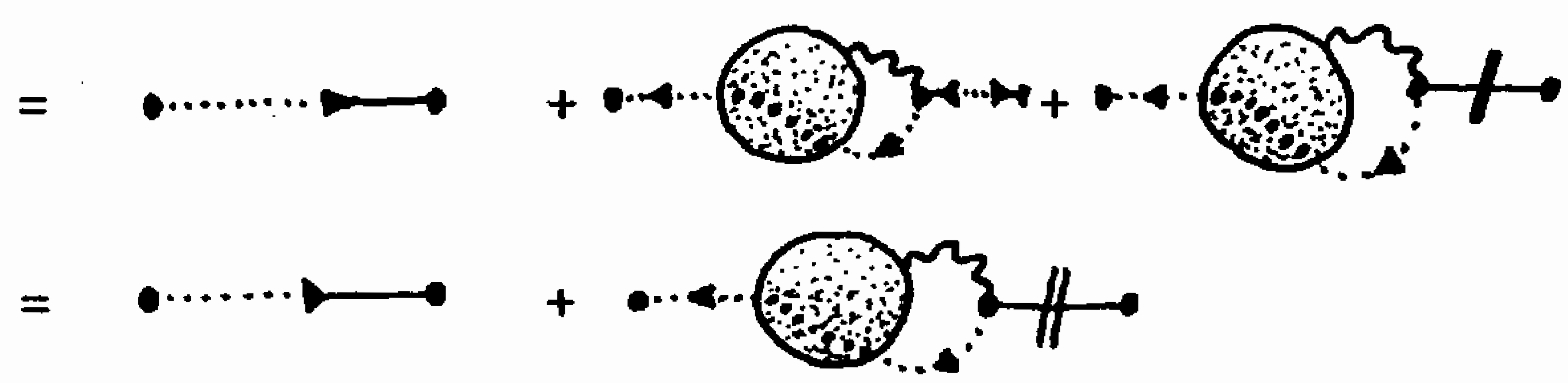


(7.13)

What happens is that as the gaugeon eats its way into a Feynman diagram, it leaves a ghost in its wake: we have indicated this by a dotted line. In QED the ghost is not coupled and Ward identities are rather simple, as in (6.9). In QCD the ghost is coupled, and the Ward identities are a more complicated affair. The simplest example is the Ward identity for the gluon self-energy:

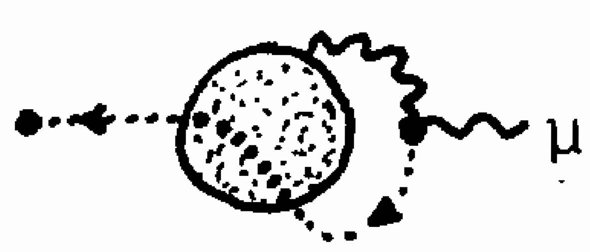
(7.14)

This takes a particularly simple form in covariant gauges, where the ghost vertex (6.41) is $h^\mu = k^\mu$. Using the ghost DS equation (7.2b) we can rewrite the above as



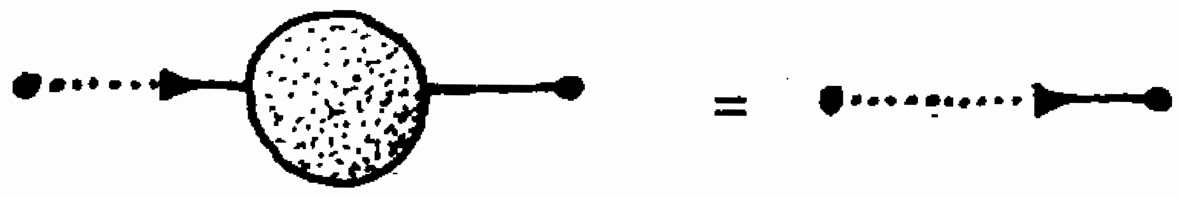
(7.15)

(we drop the vacuum bubbles). The double slash indicates the transverse projection factor $k^2 g^{\mu\nu} - k^\mu k^\nu$. As we are in the covariant gauges, the only invariant tensor with one index is k^μ, so

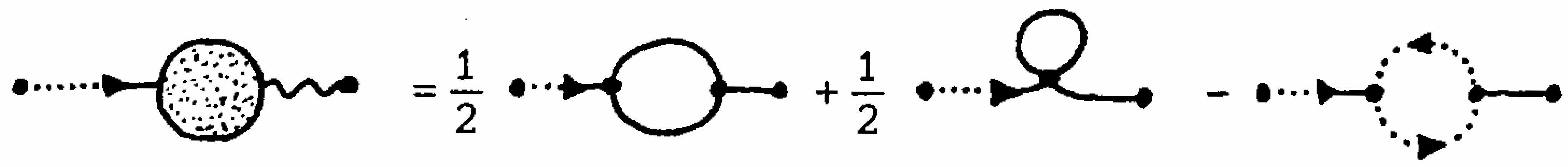


$= f(k^2)k^\mu$.

Because of the transverse projector in (7.15) such term does not contribute, and we find that the longitudinal part of the gluon propagator has no radiative corrections:

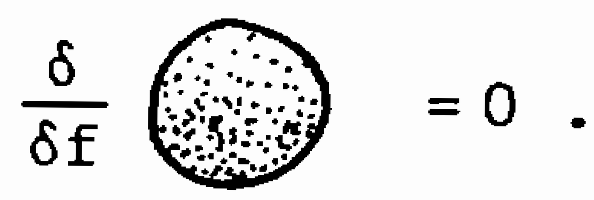
 covariant gauges (7.16)

Exercise 7.B.1 1-loop Ward identities. Check the gluon propagator at one-loop level is explicitly transverse in the Feynman gauge. Hints: Substitute diagrammatic vertices, bare Ward identities and Jacobi identities into

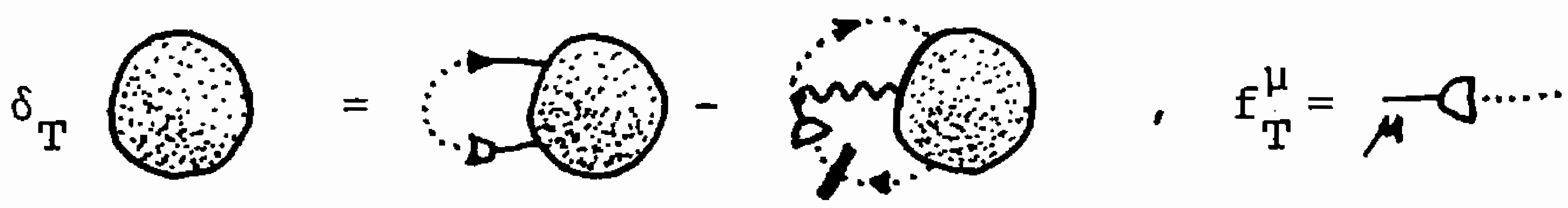


Do not drop anything because it vanishes by dimensional regularization (you are not supposed to know that yet; besides, it just messes up the proof).

Exercise 7.B.2 Prove that the vacuum bubbles are gauge invariant:



Hint: decompose f into transverse and longitudinal parts: $\delta f^\mu = \frac{(\delta f \cdot k)}{k^2} k^\mu + \delta f^\mu_T$. The ghost vertex variations are $\delta h_L = 0$, $\delta h_T = -k^2 \delta f_T$. A gauge variation of Z consists of two parts; variation of the gluon propagators and variation of the ghost vertices:



propagator variation ghost vertex variation

Exercise 7.B.3 Sign anxieties. It is pretty hard to keep track of signs in QCD; there are signs due to the antisymmetry of C_{ijk}'s, to the fermionic nature of ghosts, to momentum arrows in gluon vertices, to $-i$'s in propagators. One useful sign check is obtained by replacing full Green functions by their lowest order (tree) contributions. Check (7.13) by comparing its tree approximations to the bare vertex Ward identities of chapter 6.

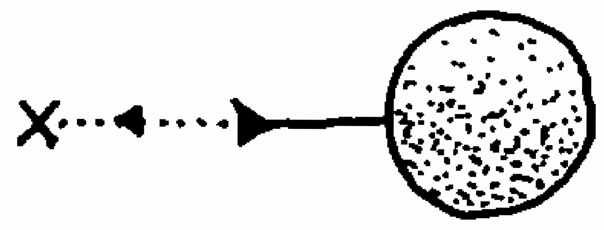
Exercise 7.B.4 Ward identities for the connected, 1PI Green functions (continuation of exercise 6.A.1). Use the relations between the full, connected and 1PI Green functions developed in chapter 2 to rewrite the Ward identities (7.12) and (7.13). Work this out for the 1PI quark vertex, gluon self-energy, etc.

C. It is upersymmetry!

The classics illustrated Ward identities (7.12) do everything we promised they would do, but Jens J. Jensen[†] is still

[†]The inventor of 3-j coefficients.

unhappy: they look different from the Ward identities in Jens' favourite textbook. What irritates Jens is the gaugeon insertion on the left-hand side of (7.12):



(7.17)

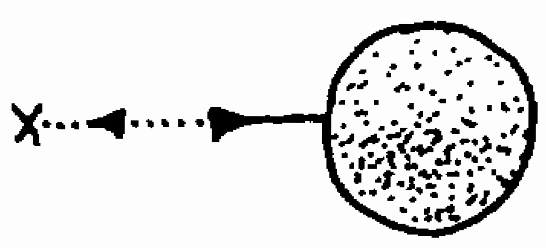
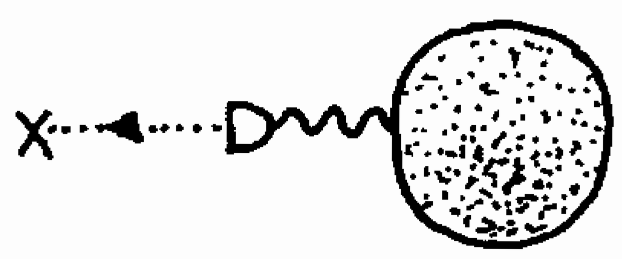
The propagator going into the blob .···· $= -ik^{\mu}/k^2$ is neither a ghost not a gluon. Well, that is no sweat. After a brief two weeks' reflection one observes that (6.61) implies

$$-\frac{ik^{\mu}}{k^2} = \frac{h_{\nu}}{B - h_T^2/k^2} D^{\nu\mu} \ . \qquad (7.18)$$

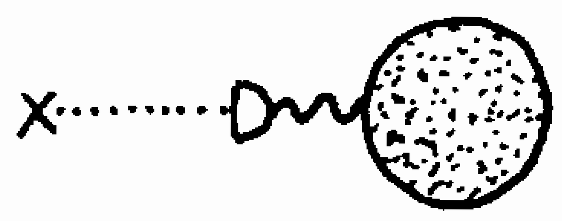
We can use this identity to replace $-k^{\mu}/k^2$ by the gluon propagator. If we introduce diagrammatic notation for the "gauge fixing functional"

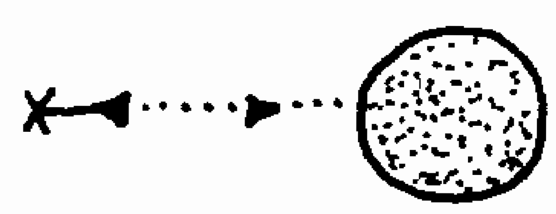
$$\frac{1}{a} \mathfrak{F}_i [A] \equiv \frac{h^{\mu}}{B - h_T^2/k^2} A_{\mu}^i =$$  (7.19)

(7.17) can be redrawn as

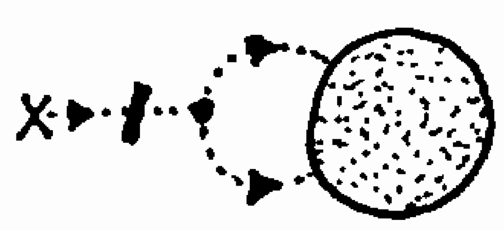
 =  (7.20)

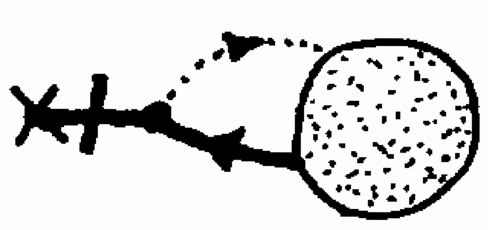
Written in the generating functional notation, the terms contributing to (7.12) are†

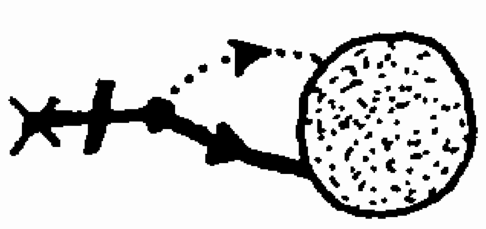
$$\int dx \xi_j (x) \frac{1}{a} F_j \left[\frac{d}{dJ(x)}\right] Z[J] =$$ 

$$\int dx J_j^{\mu}(x) \partial_{\mu} \frac{d}{d\bar{\xi}_j(x)} Z[J] =$$ 

$$\int dx J_i^{\mu}(x) \left(igC_{ijk}\right) \frac{d}{dJ_j^{\mu}(x)} \frac{d}{d\bar{\xi}_k(x)} Z[J] =$$ 

$$\int dx \bar{\xi}_j(x) \left(\frac{i}{2} gC_{ijk}\right) \frac{d}{d\bar{\xi}_j(x)} \frac{d}{d\bar{\xi}_k(x)} Z[J] = \frac{1}{2}$$ 

$$\int dx \eta^a (x) g\left(T_j\right)_a^b \frac{d}{d\eta^b(x)} \frac{d}{d\bar{\xi}_j(x)} Z[J] =$$ 

$$-\int dx \bar{\eta}_a (x) g\left(T_j\right)_b^a \frac{d}{d\bar{\eta}_b(x)} \frac{d}{d\bar{\xi}_j(x)} Z[J] = -$$  (7.21)

† No contractual obligation by Nordita regarding correctness of signs or factors of i is either expressed or implied in this or any other equation in this document.

This is as good a demonstration as any that one diagram is better than 50 symbols. In a slightly more compact notation, the Ward identities (7.12) are given functionally by

$$\left(J\cdot D\frac{d}{d\bar{\xi}} + \frac{1}{2}\bar{\xi}\cdot\left[\frac{d}{d\bar{\xi}},\frac{d}{d\bar{\xi}}\right] + \frac{1}{a}\xi\cdot F\left[\frac{d}{dJ}\right]\right)Z[J] = 0 \ . \tag{7.22}$$

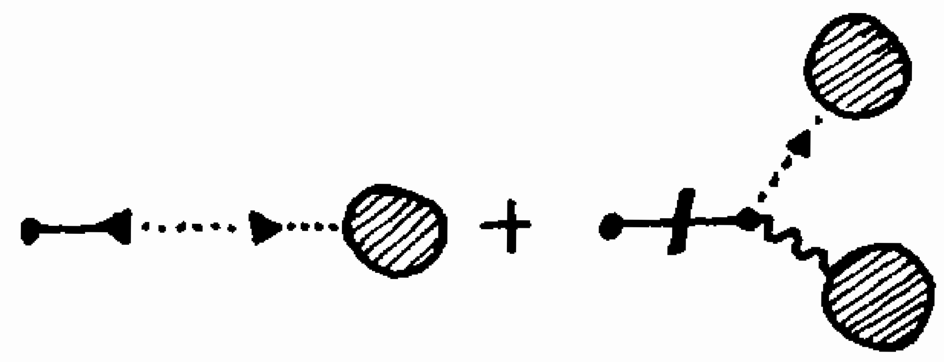
Here $D = D_\mu^{ij}$ is the covariant derivative from,(6.57), $J,\bar{\xi},\xi,\bar{\eta},\eta$ are respectively the gluon, ghost, antighost, quark, antiquark sources, and we have dropped quarks - their inclusion is straighforward.

As promised in chapter 3, the Ward identities are indeed of the form

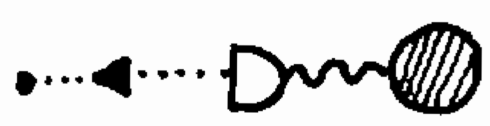
$$J_i F_i\left[\frac{d}{dJ}\right]Z[J] = 0 \quad . \tag{7.23}$$

The generators of the transformation $\delta\phi_i = \varepsilon F_i[\phi]$, equation (3.31), can be read off (7.21)

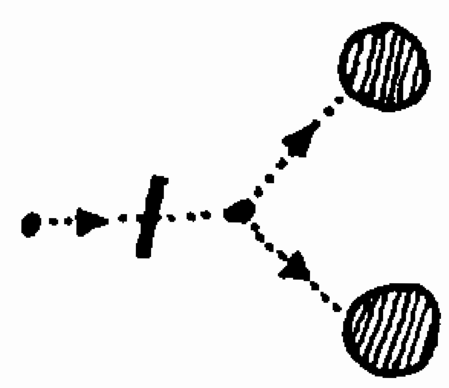
$$\delta A_\mu^i = \varepsilon D_\mu^{ij}\omega^j \quad =$$



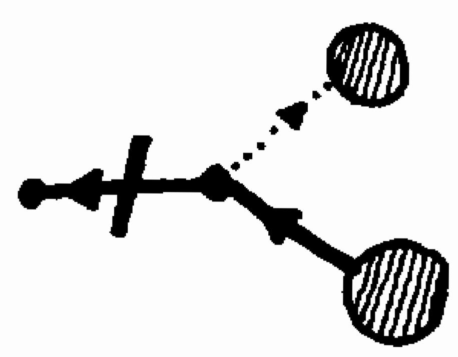
$$\delta\bar{\omega}^i = -\ \varepsilon\frac{1}{a}F_i[A] \quad =$$



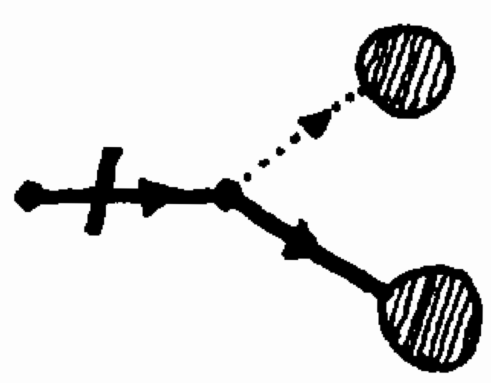
$$\delta\omega^i = -\ \varepsilon\frac{g}{2}C_{ijk}\omega^j\omega^k \quad = -\frac{1}{2}$$



$$\delta\bar{q}_a = \varepsilon ig\left(T_j\right)_a^b\omega^j\bar{q}_b \quad =$$



$$\delta q^a = -\ \varepsilon ig\left(T_j\right)_b^a\omega^j q^b \quad = - \tag{7.24}$$



According to (3.33), the action is invariant under transformations generated by $F_i[\phi]$:

$$\frac{dS[\phi]}{d\phi_i} F_i[\phi] = 0 \ .$$

This is a supersymmetry, because it mixes bosonic gluons A and fermionic ghosts $\omega, \bar{\omega}$. It is far from obvious (it was introduced by Becchi, Rouet and Stora in 1975) and it is very deep, or trivial, depending on the time of the day. In either case, the BRS symmetry is an elegant tool for proving the renormalizability of QCD, a topic that belongs to the next tome of the ultimate QCD review†. We stop here, deserting the long-leggety beasties for chaos, which, after all, is the source of all creation.

<u>Exercise 7.C.1</u> <u>BRS invariance</u>. A discouraging aspect of hidden supersymmetries like the BRS symmetry is that they are so hard to discover. QCD suggests a systematic way to construct the generators, which goes something like this:

1. Start with $\mathcal{L}_{YM}$, which is invariant under $\delta A = \varepsilon D\omega$.
2. Problem; the gluon propagator is not invertible. Break the invariance by adding $\mathcal{L}_{fix} = -(\partial\cdot A)^2/(2a)$. This generates
$$\delta\mathcal{L}_{fix} = -\frac{\varepsilon}{a}(\partial\cdot A)(\partial\cdot D)\omega \ .$$
3. Attempt to restore the symmetry by adding a new field with variation
$$\delta\bar{\omega} = \varepsilon(\partial\cdot A)/a$$
and action term
$$\mathcal{L}_{ghost} = \bar{\omega}(\partial\cdot D)\omega \ .$$
4. This does not quite work because D is field-dependent, and $\delta\mathcal{L}_{ghost}$ generates an extra term
$$\bar{\omega}_i C_{ijk}\partial\cdot D_{k\ell}\omega_\ell\omega_j \ .$$
5. Save the day by varying ω as well
$$\delta\omega_i = -\frac{\varepsilon}{2}C_{ijk}\omega_j\omega_k .$$

Antisymmetry of C_{ijk} forces you to take ω fermionic. Check all steps in the above argument.

<u>Exercise 7.C.2</u> <u>Ward identities for the effective action</u>. Use the methods of chapter 2 to rewrite (7.22) in terms of 1PI functionals. Hint: introduce extra sources for the non-linear terms in (7.24).

† A.D. Kennedy, in preparation.