5. SPACETIME PROPAGATION

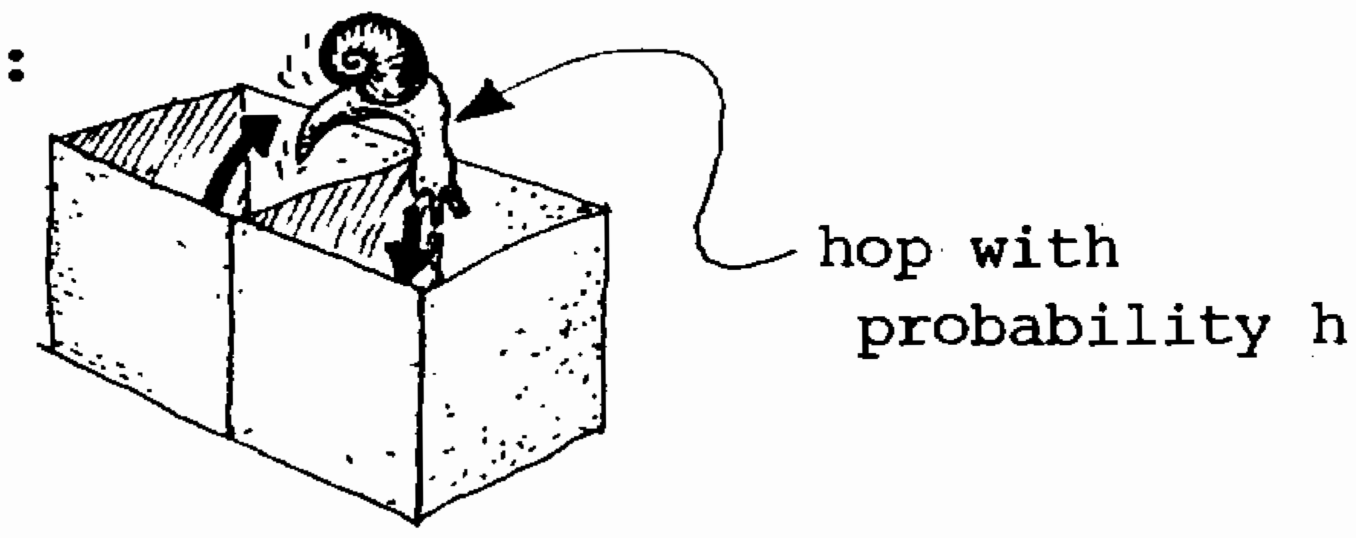
Until now the collective indices have stood for all particle labels; spacetime location, spin, particle type and so on. To apply field theory to particle physics we have to describe propagation of particles through the spacetime. I find it most convenient to formulate the field theory in our spacetime as an analytic continuation from a Euclidean world in which there is no distinction between time and space. What do we mean by propagation in such a space?

Our formulation is inevitably phenomenological: we have no idea what the structure of our spacetime on distances much shorter than nuclear sizes might be. The spacetime might be discrete rather than continuous, or it might have geometry different from the one we observe at the accessible distance scales. The formalism we use should reflect this ignorance. We will deal with this problem by subdividing the space into small cells and requiring that our theory be insensitive to distances comparable to or smaller than the cell sizes.

Our next problem is that we have no idea why there are particles, and why or how they propagate. The most we can say is that there is some probability that a particle hops from one spacetime cell to another spacetime cell. At the beginning of the century, the discovery of Brownian motion showed that matter was not continuous but was made up of atoms. In particle physics we have no indication of having reached the distance scales in which any new spacetime structure is being sensed: hence for us this hopping probability has no direct physical significance. It is simply a phenomenological parameter: in the continuum limit it will be replaced by the mass of the particle.

A. Free propagation

We assume for the time being that the state of a particle is specified by its spacetime position, and that it has no further labels (such as spin or color): $i = (x_1, x_2, \ldots, x_d)$. What is it like to be free? A free particle exists only in itself and for itself; it neither sees nor feels the others; it is, in this chilly sense, free. But if it is not at once paralyzed by the vast possibilities opened to it, it soon becomes perplexed by the problems of realizing any of them alone. Born free, it is constrained by the very lack of constraint. Sitting in its cell, it is faced by a choice of doing nothing (s = stopping probability) or hopping into any of the $2d$ neighboring cells (h = hopping probability):

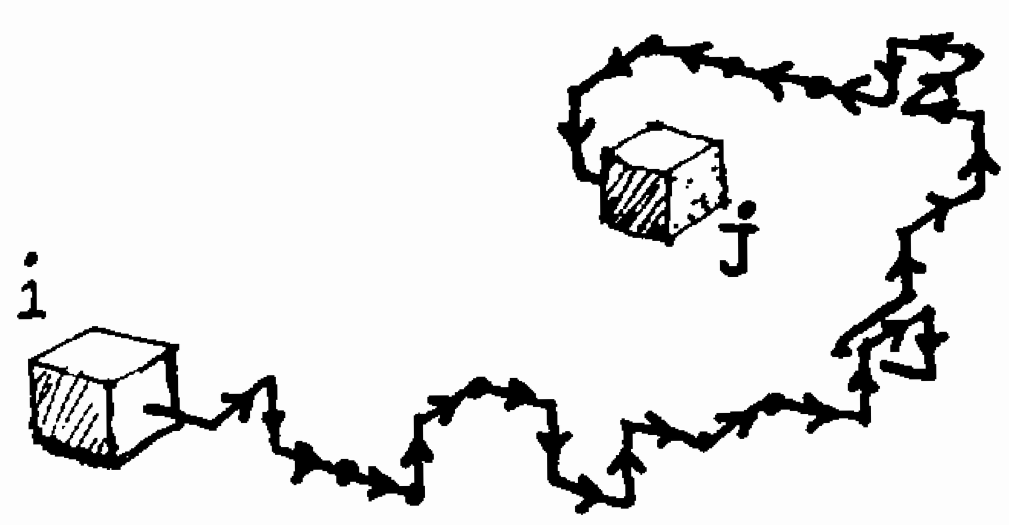


The number of neighboring cells defines, if you wish, the dimension of the spacetime. The hopping and stopping probabilities are related by the probability conservation: $1 = s + 2dh$. Taking the hopping probability to be the same in all directions means that we have assumed that the space is isotropic.

Our next assumption is that the spacetime is homogeneous, i.e. that the hopping probability does not depend on the location of the cell. (Otherwise the propagation is not free, but is constrained by some external geometry.) This can either mean that the spacetime is infinite, or that it is compact and periodic (a torus). That is again something beyond our ken - we proceed in the hope that the predictions of our theory will be insensitive to very large distances.

The isotropy and homogeneity assumptions imply that our theory should be invariant under rotations and translations. The requirement of insensitivity to the very short and very long distances means that the theory must have nice ultraviolet and infrared properties.

A particle can start in a spacetime cell i and hop along until it stops in the cell j. The probability of this process is $h^L s$, where L is the number of steps in the corresponding path:



The total probability that a particle wanders from the i-th cell and stops in the j-th cell is the sum of probabilities associated with all possible paths connecting the two cells:

$$\Delta_{ij} = s \sum_L h^L N_{ij}(L) \ . \tag{5.1}$$

$N_{ij}(L)$ is the number of all paths of length L connecting i and j. Define a stepping matrix

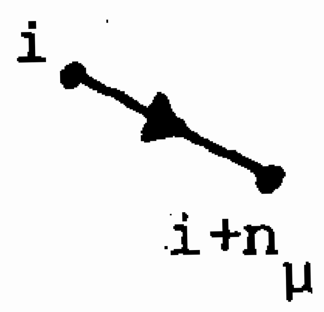
$$(S^\mu)_{ij} = \delta_{i+n_\mu, j} \ . \tag{5.2}$$

If a particle is introduced into the i-th cell by a source

$$J_k = \delta_{ik} \ ,$$

the stepping matrix moves it into a neighboring cell:

$$(S^\mu J)_k = \delta_{i+n_\mu, k} \rightarrow$$



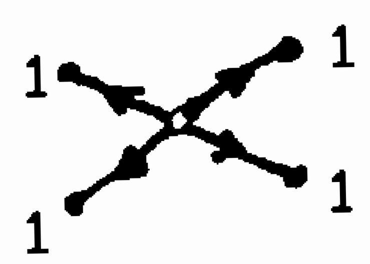
The operator

$$(h\cdot S)_{ij} = \sum_{\mu=1}^{d} h_\mu [(S^\mu)_{ij} + (S^\mu)_{ji}] \ ,$$

$$h_\mu = (h,h,\ldots,h) \tag{5.3}$$

generates all paths of length 1 with probability h:

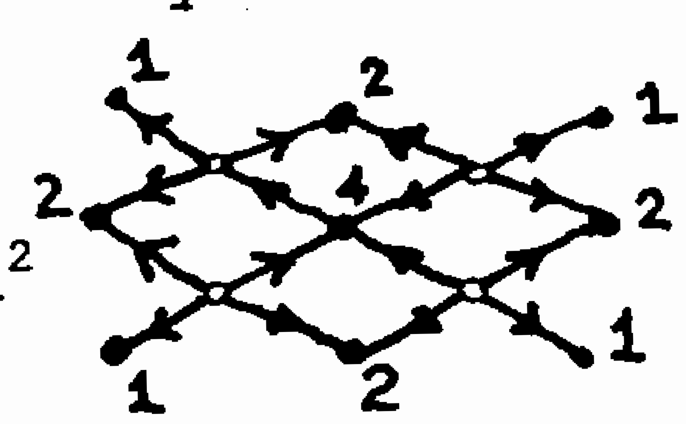
$$(h\cdot S)J = h$$



i-th cell

(The examples are drawn in two dimensions). The paths of length 2 are generated by

$$(h\cdot S)^2 J = h^2$$



and so on. Note that the k-th component of the vector $(h\cdot S)^L J$ counts the number of paths of length L connecting the i-th and the k-th spacetime cells. The total probability that the particle stops in the k-th cell is given by

$$\phi_k = s \sum_L (h\cdot S)^L_{ki} J_i \quad ,$$

$$\phi = \frac{s}{1-h\cdot S} J \quad . \tag{5.4}$$

The value of the field[†] ϕ_k at a spacetime point k measures the probability of observing the particle introduced into the system by the source J. The Euclidean free scalar particle propagator (5.1) is given by

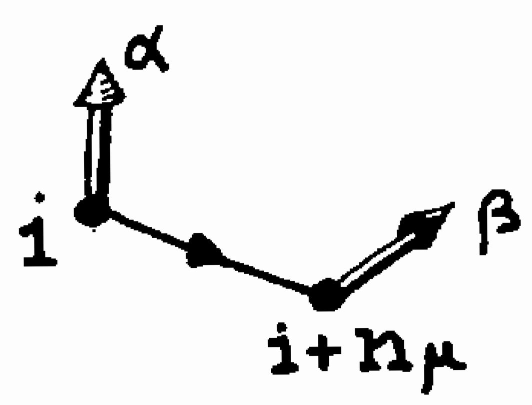
$$\Delta_{ij} = \left(\frac{s}{1-h\cdot S}\right)_{ij} \quad , \tag{5.5}$$

or, in the continuum limit (do exercise 5.A.1) by

$$\Delta(x,y) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik\cdot(x-y)}}{k^2+m^2} \quad . \tag{5.6}$$

So far we have assumed that the particle hops to any neighboring cell with the same probability. What happens if the particle hiding in the spacetime cell is not a small spherical object, but something long and shapely? In that case, we have to introduce spin labels to define the particle orientation: $i = (x_\mu, \alpha)$. Such a particle will hop and retain its orientation with some probability, and hop and change its orientation with a different probability. The hopping probability h is now replaced by a hopping matrix

$$(h^\mu)_{\alpha\beta} = \tag{5.7}$$



which describes the probability that a particle with the spin label α hops one step in the direction μ and flips its spin to β. We do not want to give up the isotropy and homogeneity of spacetime, so the hopping matrix can depend only on the relative orientations of the two spins. In other words, the hopping matrix must be an invariant tensor under spacetime translations and rotations.

[†] Interpreting ϕ as a field is consistent with the previous definition of a free field, equations (2.22) and (2.25).

How does one describe orientation of a particle? That depends on the particle type. For example, if the particle orientation can be specified by a d-dimensional vector, we need d spin labels. We shall always assume that the range of the spin index is finite. In the language of group theory this means that we shall consider only the finite dimensional representations of the rotation group. Furthermore, we shall be interested only in irreducible representations. The physical reason is that reducible representations are resolved into irreducible components by quantum corrections. For example, if a free propagator contains both an isotropic part which propagates as a scalar (5.5) and a non-isotropic remainder, one-loop corrections will be in general different for the two parts.

If a particle of spin α is introduced into i-th cell by means of a source

$$J_{k\beta} = \delta_{\alpha\beta}\delta_{ik} \quad ,$$

the stepping matrix (5.2) generates the probabilities associated with all paths of length one:

$$(h\cdot S)J_{k\beta} = h^{\mu}_{\beta\gamma}\left(S^{\mu}_{k\ell} + S^{\mu}_{\ell k}\right)J_{\ell\gamma} \quad .$$

The probabilities associated with all paths of length two are given by $(h\cdot S)^2 J$, and so on. Hence the propagator for a free spinning particle is given by

$$\begin{aligned}\Delta_{i\alpha,j\beta} &= s\delta_{ij}\delta_{\alpha\beta} + s\sum_{L>0}(h\cdot S)^{L}_{i\alpha,j\beta} \\ &= \left(\frac{s}{1-(h\cdot S)}\right)_{i\alpha,j\beta} \quad . \qquad (5.8)\end{aligned}$$

To make further headway, one has to be more specific about the hopping probability h^{μ}. This would get us too deep into group theory, and (if we started thinking about fermions), lead to ulcers. We stop now.

Exercise 5.A.1 Continuum propagator. Define the finite difference operator by

$$\partial f(x) = \frac{f(x+\frac{a}{2}) - f(x-\frac{a}{2})}{a}$$

where a is the lattice spacing. Show that

$$\frac{1}{h}\sum^{d} h_\mu (S^\mu_{ij} + S^\mu_{ji}) = 2d + a^2\partial^2 \quad ,$$

where $\partial^2 = \partial_\mu\partial_\mu$ is the finite difference Laplacian. Show that the Euclidean scalar lattice propagator (5.5) is given by

$$\Delta^{-1}_{ij} = 1 - \frac{ha^2}{s}\partial^2 .$$

The mass in the continuum propagator (5.6) is related to the hopping parameter by

$$m^2 = \frac{s}{ha^2} \; . \qquad (5.9)$$

If the particle does not like hopping $(h \to 0)$, the mass is infinite and there is no propagation. If the particle does not like stopping $(s \to 0)$, the mass is zero and the particle zips all over the space.
Diagonalize ∂^2 by Fourier transforming and derive (5.6).

B. A leap of faith

We have constructed the Euclidean free-particle propagator from a few basic notions such as addition of probabilities and spacetime homogeneity and isotropy. At some point we have to face two non-intuitive facts: our world is Minkowskian, not Euclidean, and the theory of elementary particles is quantum mechanics, not statistical mechanics. Usually somebody tells you that the quantum mechanics is obtained from the classical mechanics by replacing Poisson brackets by commutators (canonical quantization). This gives me no intuition about quantum mechanics. With my present (lack of) understanding, I find it easier to think of field theory in terms of probabilities, as we have done up to now, and then make a leap of faith by saying: our world is a Wick rotation of the Euclidean world,

$$x_4 = ix_0 \; . \qquad (5.10)$$

This gives us

1) special relativity $\qquad g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \qquad (5.11)$

2) quantum mechanics; Boltzmann weight e^S is replaced by a phase factor $e^{iS/\hbar}$.

For example, Euclidean action

$$S[\phi] = \int d^dx \tfrac{1}{2}\phi(x)\,(\partial^2 + m^2)\,\phi(x) \quad , \tag{5.12}$$

is replaced by the Minkowski action

$$\frac{i}{\hbar}\, S[\phi] = \frac{i}{\hbar}\int d^dx \tfrac{1}{2}\phi(x)\,(g_{\mu\nu}\partial^\mu\partial^\nu + m^2)\,\phi(x) \quad , \tag{5.13}$$

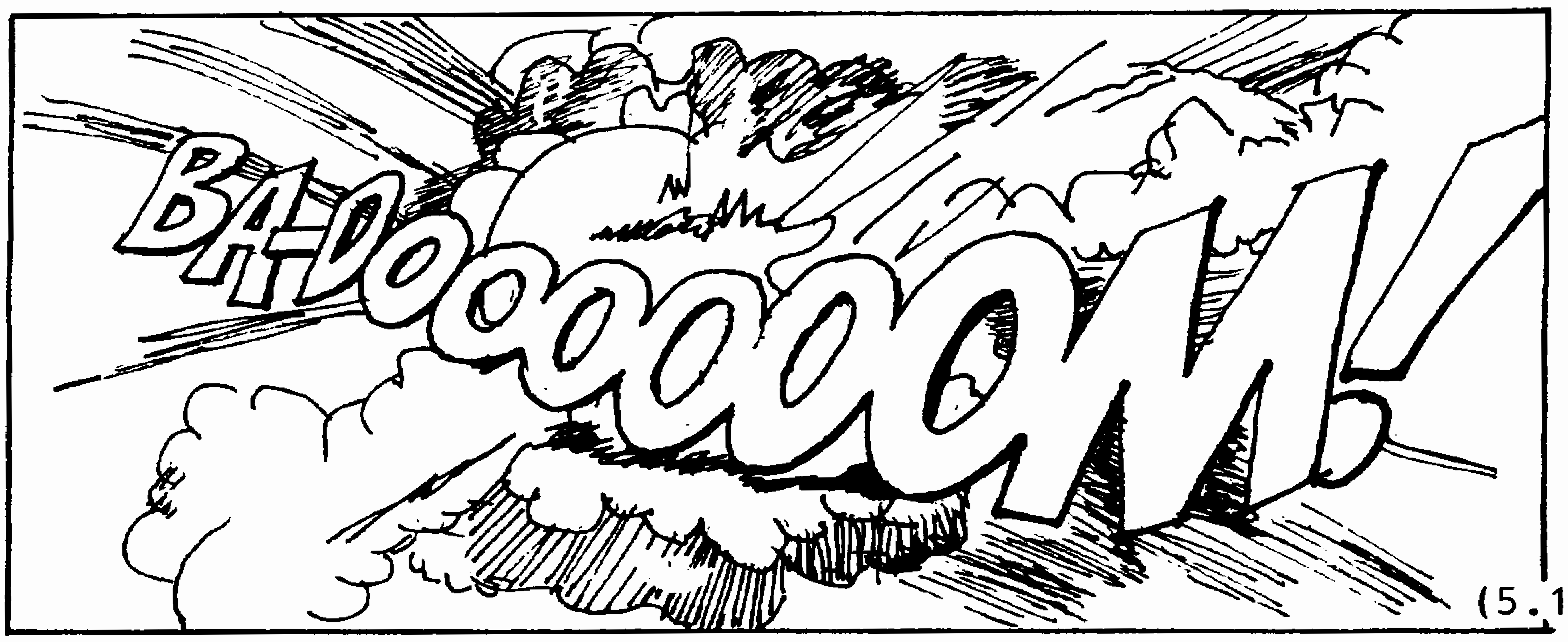
where the imaginary factor i is the jacobian from the change of variables (5.10).

3) correspondence principle; Planck constant $\hbar$ is the scale of quantum fluctuations, and the classical mechanics is the large action limit of the quantum theory.

It is not good enough†, but it will get us through the night.

C. Scattering matrix

A run-of-the-mill particle scattering experiment looks something like this



(5.14)

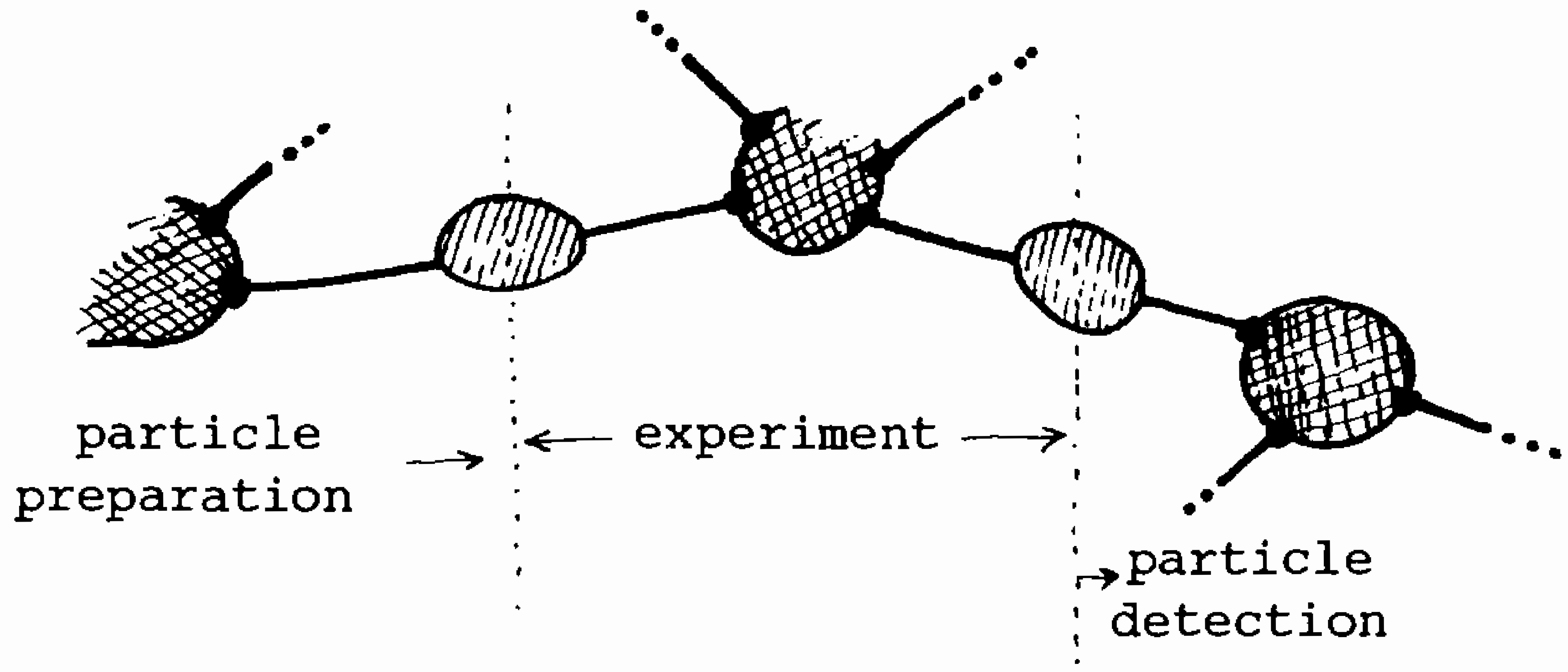
Particles with sharply defined 4-momentum are accelerated over kilometer distances, collide in regions of nuclear size and the

---

†There is a little problem with interpreting measurements.

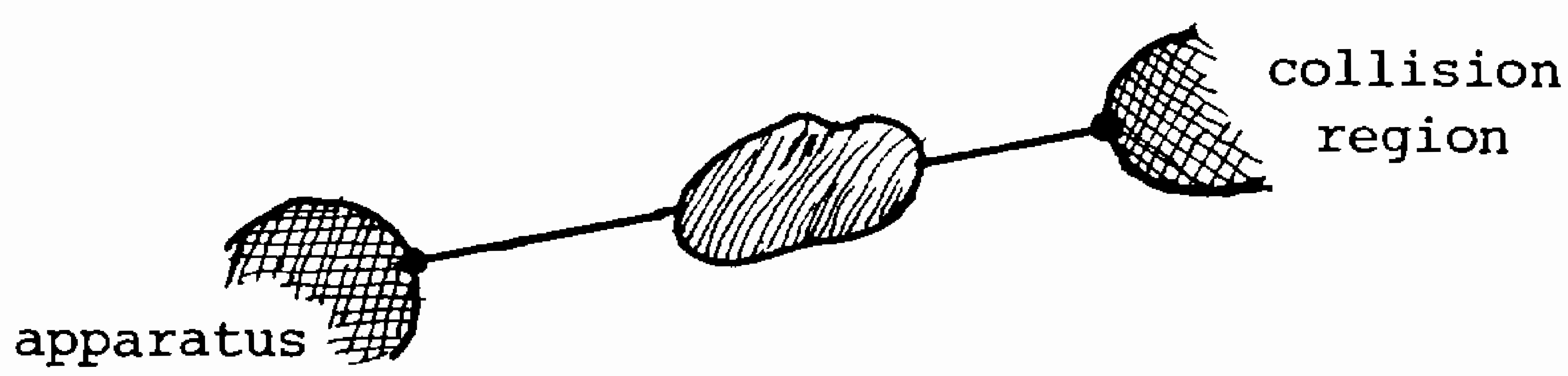
resulting particles fly tens of meters to detectors. The theoretical predictions for such experiments are expressed in terms of connected Green functions. If you think about it, you will realize that the experiments measure the effective vertices, or the 1PI Green functions.

If you <u>really</u> think about it, our formulation in terms of sources is a brave idealization. In reality the entire experiment is one large system



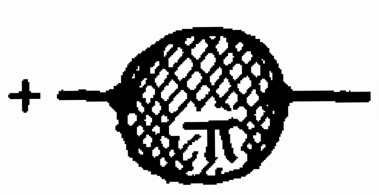
(5.15)

and approximating the experimental apparatus by sources makes sense only when the interaction region can be well separated. The particles which traverse the macroscopic distances between the interaction region and the experimental apparatus are classical, mass-shell particles with $k^2 = m^2$:

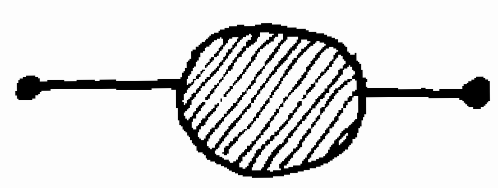


(5.16)

We can measure the mass of these particles by measuring their four-momenta. The theory predicts a mass-shift

$m^2 = m_0^2 +$  $\Big|_{k^2 = m^2}$ (5.17)

This relates the bare mass (mass with all interactions turned off) to the physical mass. The theory also predicts a wave-function renormalization

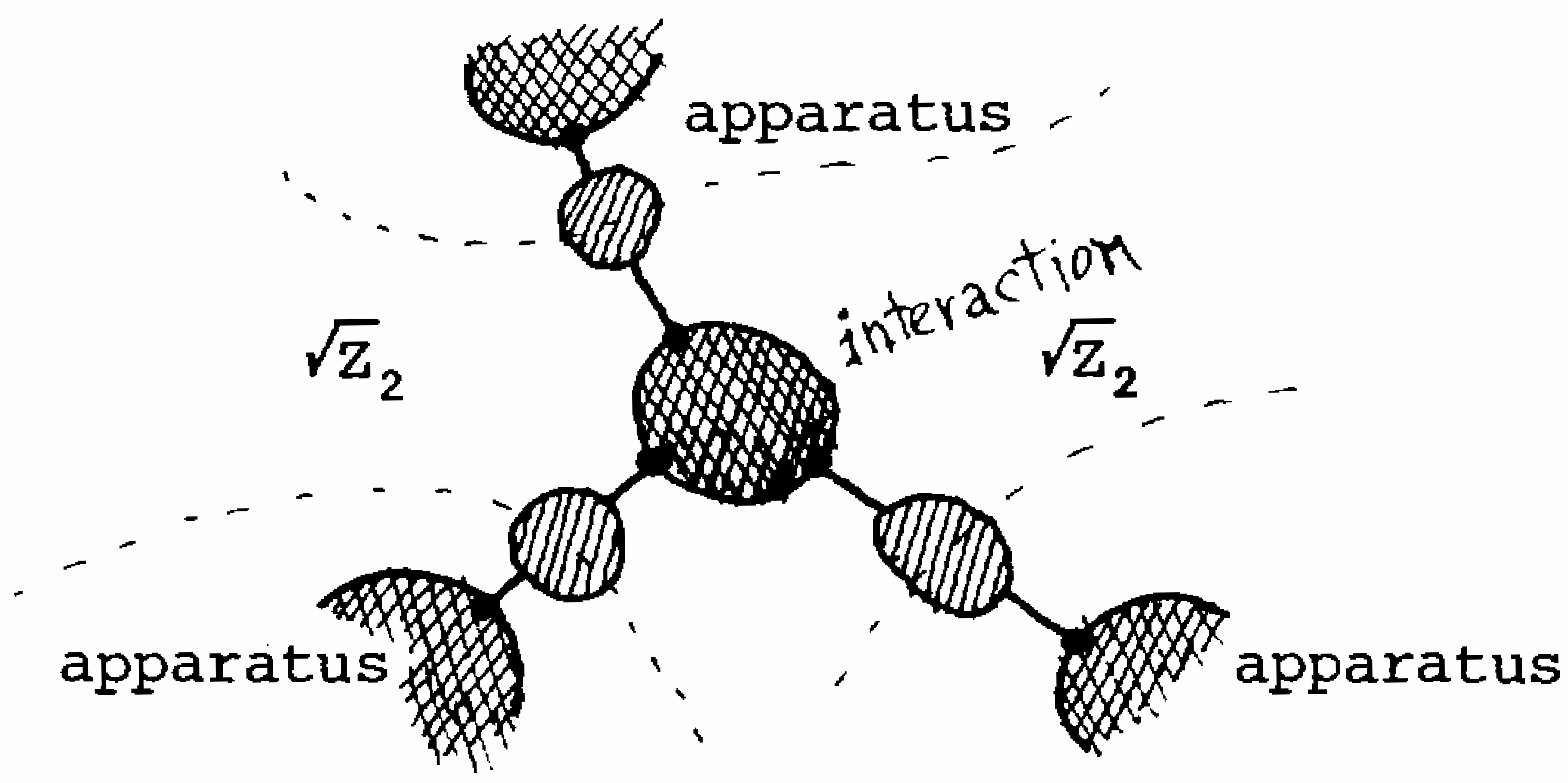
 $\Big|_{k^2 = m^2} = \dfrac{Z_2}{k^2 - m^2}$. (5.18)

If the particles also carry spin, there will be further mass-shell constraints. They are expressed in terms of polarizations $\varepsilon^\mu(k)$, spinor wave functions $u_\alpha(k)$, etc.; we shall soon see such objects. They are the reason why Z_2 is called the "wave function renormalization constant".

A connected Green function (2.17) has a propagator on each external leg. These propagators develop poles if the corresponding particles traverse macroscopic distances, and what is probed in an experiment is not the entire Green function, but only its mass-shell amputation

$$\prod_i (k_i^2 - m_i^2) G^{(c)}(k_1, k_2, \dots)\big|_{k^2=m^2} \quad .$$

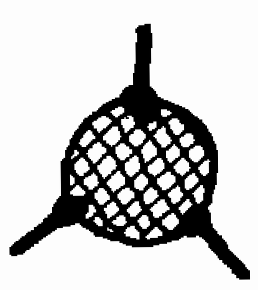
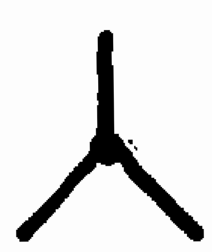
The renormalization constants Z_2 survive all such amputations, and cannot be disentangled from the measurements of the physical coupling constants:



The resolution of this problem is to absorb Z_2 into the definitions of the physical coupling constants by

$$g = Z_2^{k/2} Z_1^{-1} g_0 \quad , \tag{5.19}$$

where g_0 is the bare coupling constant (for a vertex with k legs), and the vertex renormalizations Z_1 are computed from

 $\Big|_{k^2=m^2} \equiv \frac{1}{Z_1}$  (5.20)

(and so on for higher vertices). The wave function renormalizations contribute factors of $\sqrt{Z_2}$ because they must be shared in a sisterly fashion between the two ends of each propagator. So, the quantities that are really measured in experiments, and therefore called the S-matrix (scattering matrix) elements, are

$$S(k_1,k_2,\ldots) = \prod_i \frac{k_i^2 - m_i^2}{\sqrt{Z_{2,i}}} G^{(c)}(k_1,k_2,\ldots)\Big|_{\text{mass-shell}} , \qquad (5.21)$$

(for particles with spin we should also add polarization wave functions on the external legs). Here the $Z_2^{-\frac{1}{2}}$ factors account for the bits of renormalization constants absorbed by the experimental apparatus, and the bare masses and couplings are to be re-expressed in terms of the physical ones by (5.17) and (5.19).

This is called <u>renormalization</u>. It is not here because of (possible) ultraviolet divergences, but because it is inevitable. The only way to compare our theory with nature is to relate our Green functions to physically measurable parameters, and then re-express all predictions of the theory in terms of those parameters.

Renormalization should not be confused with regularization. Regularization is a mathematical problem of defining infinite sums in the intermediate steps of field theory calculations; renormalization is a unique, physically determined procedure of expressing the physical predictions of a theory in terms of physically measurable parameters.