

(replacement)

## Exercise 5.A.1    Diagonalization of propagators

Consider the stepping matrix (5.2) for a  $n$ -site lattice with periodic boundary conditions (assume 1-dimensional, for now):

$$S = \begin{pmatrix} 0 & 1 & \cdots & \cdots \\ \vdots & 0 & 1 & \cdots \\ \vdots & \vdots & 0 & 1 \\ \vdots & \vdots & \vdots & 0 \\ 1 & \cdots & \cdots & 0 \end{pmatrix}$$

$S$  satisfies  $S^n = 1$ , with eigenvalues  $\lambda_r = e^{i2\pi \frac{r}{n}}$ .  $S$  can be diagonalized by

$$S = C \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} C^\dagger$$

$$C_{jr} = \frac{1}{\sqrt{n}} e^{i2\pi \frac{rj}{n}}, \quad (C^\dagger)_{rj} = C_{jr}^*$$

check that

$$S_{ij} = \sum_r C_{ir} \lambda_r C_{rj}^\dagger = \delta_{i+1, j}$$

Now the propagator (5.5) can be diagonalized

$$\Delta_{ij} = \sum_r C_{ir} \frac{s}{1 - h(\lambda_r + \lambda_r^*)} C_{rj}^\dagger = \frac{s}{n} \sum_r \frac{e^{i2\pi r \frac{i-j}{n}}}{1 - 2h \cos(2\pi \frac{r}{n})}$$

This is the lattice propagator. To take its continuum limit, introduce lattice spacing, and <sup>the</sup> continuum length and momentum variables:

$$\text{lattice spacing } a = \frac{L}{n} \quad \leftarrow \text{arbitrary length scale}$$

$$\text{momentum } k = \frac{2\pi r}{L}; \quad \text{coordinates } x = ia, y = ja, \dots$$

For small momenta (i.e. distances much larger than the lattice spacing)

$$\Delta(x, y) = a s \int \frac{dk}{2\pi} \frac{e^{ik(x-y)}}{1-2h + \hbar a^2 k^2 + \dots}$$

By <sup>the</sup> probability conservation  $1-2h = s$ . We replace the hopping parameter by the mass parameter

$$m^2 = \frac{s}{\hbar a^2}$$

If the particle does not like hopping ( $h \rightarrow 0$ ), the mass is infinite, and there is no propagation. If the particle does not like stopping ( $s \rightarrow 0$ ), the mass is zero. The argument is the same for arbitrary dimension, and it yields the Euclidean continuum propagator:

$$\Delta(x, y) = m^2 a^d \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik \cdot (x-y)}}{m^2 + k^2}$$

(The factor  $a^d m^2$  is usually absorbed into the definition of  $\Delta(x, y)$ ).

Exercise 5.A.2 Probability conservation. Check that  $\Delta(x, y)$  satisfies probability conservation:  $\int dy \Delta(x, y) = 1$ .

Exercise 5.A.3 Massless propagators. Show

that for a massless particle the continuum propagator is given by

$$\Delta(x, y) = \frac{\Gamma(\frac{d}{2}-1)}{4\pi^{d/2}} \frac{1}{|x-y|^{d-2}}$$

Can you derive this by a Euclidean random walk argument?

Exercise 5.A.4 Fermion propagators. Random walk now has a tough constraint: the walks must be non-self-intersecting. Show that by taking into account vacuum fluctuations you get the same propagator as in the bose case.

Exercise 5.A.5 Spinning particles. Construct propagators for (massive) spin-1 and spin  $1/2$  particles.

Exercise 5.A.6 Derive spin-statistics theorem; probability conservation requires integer spins to be bose particles, half-integer spins to be fermi particles.

Exercise 5.A.7 Pull out the propagator from a different hat (suggestions: stochastic quantization; microcanonical field theory; heat-kernel formulation).

Exercise 5.A.8 Construct propagator for some non-trivial geometry, such as random walk on a non-abelian Lie group manifold.

and exercise on boundary conditions. but:  $S^k$  can't really, for  $k=1$  (which is for torus and plane, hence connections of order  $\hbar^2$ ). by prob conservation  $\hbar \leq \epsilon t$ , thus  $\epsilon$  drops like  $\hbar^2$  if needed.