4. FERMIONS

A. Pauli principle

In chapter 2 we have assumed that the Green functions are symmetric, i.e. that the particles we are describing are bose particles. What happens if the Pauli principle is at work? The Pauli principle is the quantum mechanical version of Archimedes' law. Archimedes' law says that two bodies cannot be in the same place at the same time; the Pauli principle does not allow existence of more than one particle in a given quantum-mechanical state.

In the Green function formalism the state of a particle is specified by its collective index (particle type, spin, position ...). Take a source which produces a particle in a definite quantum-mechanical state, i.e. a source which is nonvanishing only for one value of the collective index:

$$J_i = \delta_{im} \quad .$$

If the Pauli principle is at work, the Green functions must vanish any time two or more of their indices take the same value:

$$G_{ijk..\ell} J_i J_j = 0 \quad . \tag{4.1}$$

The basic assumption of the whole scheme that we are expounding here is that the amplitudes are additive. A linear superposition of state is also a state, and it too must satisfy the Pauli principle (here $K_i = \delta_{i\ell}$ is a source for a particle in state $\underline{\ell}$):

$$G_{ijk..}(J_i + K_i)(J_j + K_j) = 0$$

$$\Rightarrow (G_{ijk..} + G_{jik..}) J_i K_j$$

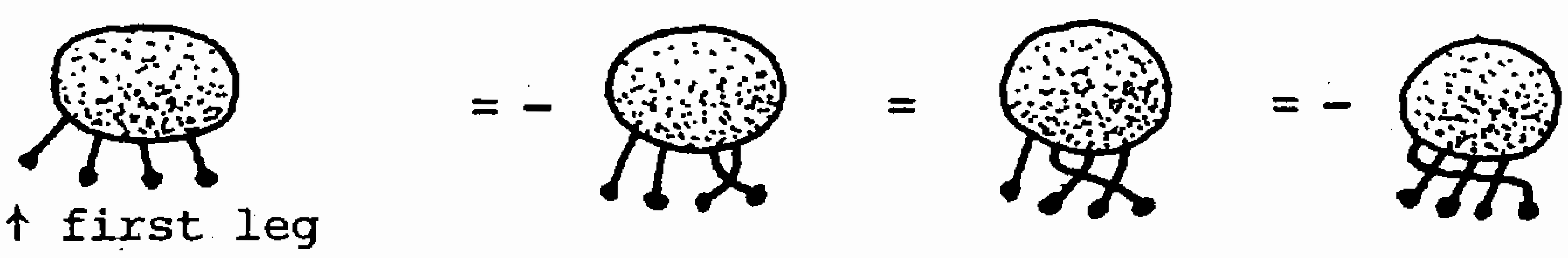
Consequently, the Green functions must be antisymmetric under interchange of fermionic indices:

$$G_{ijk...} = - G_{jik...} = G_{jki...} = \ldots \quad . \tag{4.2}$$

(In the compact index notation a multiplet can include both bosons and fermions: for example, for QED (cf. equation (3.27)) $\phi_i = (\psi,\bar{\psi},A_\mu)$ stands for electrons, positrons and photons. In such cases we have to distinguish between the fermionic and the bosonic indices.)

From now on I will consider only the theories in which all Green functions have even numbers of fermionic legs. Another way of saying this is that we shall always assume that the action is a commuting number.

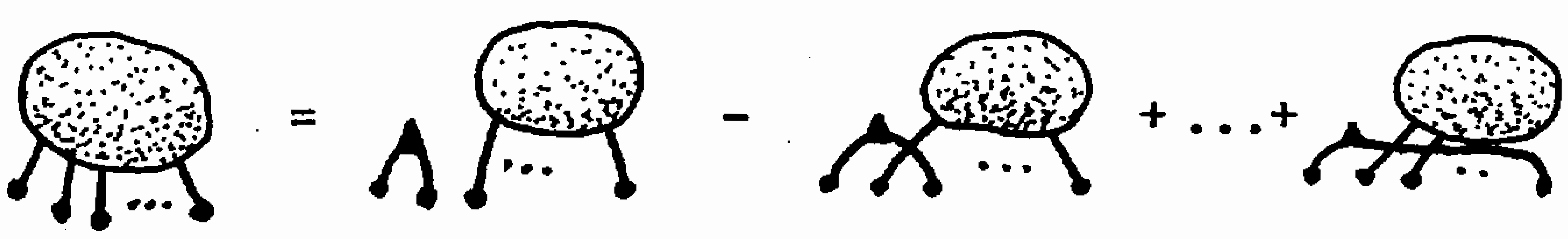
Fermionic Green functions with even numbers of legs are anti-cyclic:



$$\langle\psi_i\psi_j\psi_k\psi_\ell\rangle = -\langle\psi_i\psi_j\psi_\ell\psi_k\rangle = \langle\psi_i\psi_\ell\psi_j\psi_k\rangle = -\langle\psi_\ell\psi_i\psi_j\psi_k\rangle \quad . \tag{4.3}$$

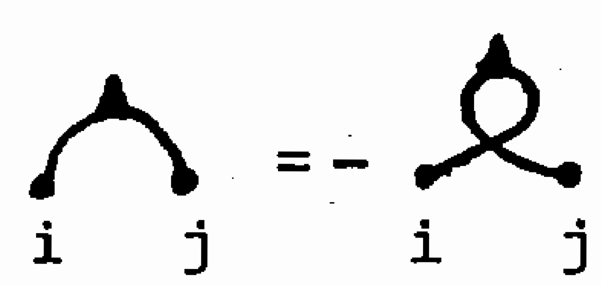
In order to keep track of signs, the diagrammatic notation must indicate which leg is the first leg. We do it by always drawing the fermionic legs below the Green function blobs, and taking the leftmost leg to be the first one. This fixes all relative signs. The overall sign is physically irrelevant.

The perturbation expansion can be generated by the Dyson-Schwinger equations, just as in the bosonic case. The diagrams and the combinatoric factors are the same; the only difference is the signs due to the antisymmetry of Green functions. For example, the free fermion field theory DS equations are



(4.4)

Fermionic propagators are antisymmetric, so the first and the second legs must be distinguished. We do this diagrammatically by drawing a little wart on the propagator:

$\Delta_{ij} =$ 

$= -\Delta_{ji}$. (4.5)

Exercise 4.A.1 Can you prove that fermionic Green functions must have an even number of legs?

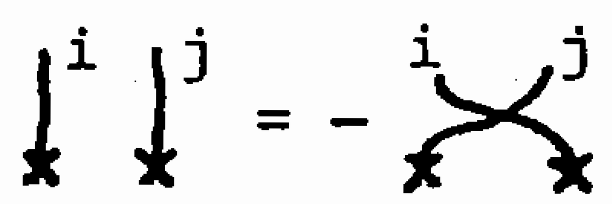
Exercise 4.A.2 Can you prove that fermionic Green functions need not have an even number of legs?

B. Anticommuting sources

In the bosonic case, the discussion of the general properties of Green functions was greatly facilitated by the introduction of generating functionals. In the fermionic case we cannot simply add scalar source functions (2.4) and form the vacuum Green function (2.10), because this would yield zero, identically:

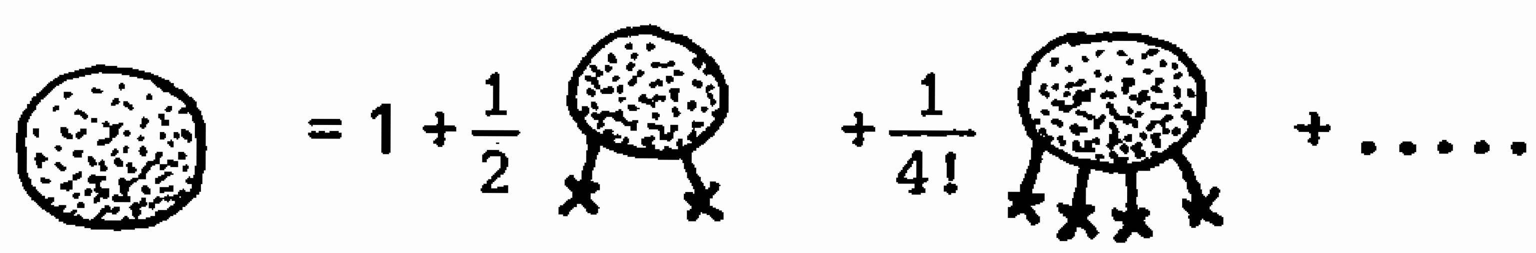
$$G_{ijk\ldots}J_iJ_jJ_k\ldots = G_{ijk\ldots}\frac{1}{2}(J_iJ_j - J_jJ_i)J_k = 0 \quad .$$

However, a simple trick provides a way out; we replace J_i by anticommuting sources:



$$\eta_i\eta_j = -\eta_j\eta_i \quad . \tag{4.6}$$

Then the fermionic generating functional can be defined as



$$Z[\eta] = \sum_{m=0}\frac{1}{(2m)!}G_{ij..k}\eta_k\ldots\eta_j\eta_i \quad . \tag{4.7}$$

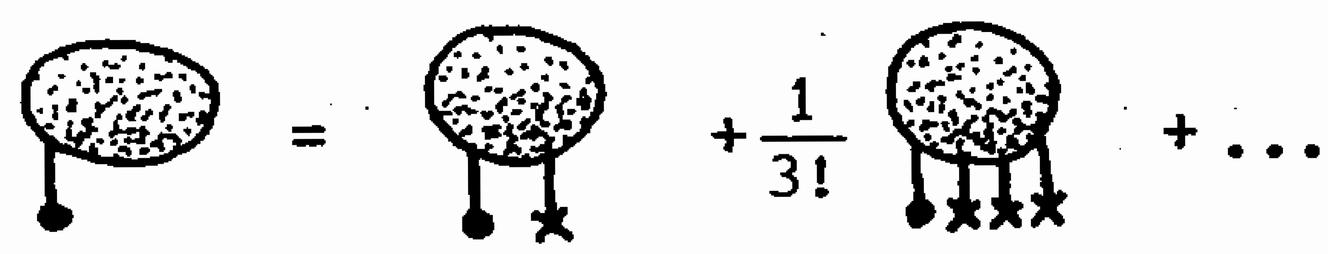
(Remember, our Green functions always have even numbers of legs.)

The signs due to sources are kept track of by drawing the sources ordered along the bottom of the diagram. Green functions can be retrieved from the generating functional by differentiation, just as in the bosonic case (2.11). However, the derivatives must also be anticommuting:

$$\frac{d}{d\eta_i}\eta_j = \delta_{ij} - \eta_j\frac{d}{d\eta_i} \quad ,$$

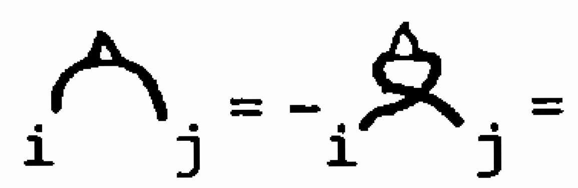
$$\frac{d}{d\eta_i}\frac{d}{d\eta_j} = -\frac{d}{d\eta_j}\frac{d}{d\eta_i} \quad . \tag{4.8}$$

All the relations between the full, connected and 1PI generating functionals that we have derived for the bosonic case take the same form for the fermionic generating functionals. There is only one sign subtlety. As all the terms in (4.7) involve even numbers of sources, all generating functionals are commuting numbers, and the sources implicit in them lead to no sign confusion. However, if a leg is pulled out by differentiation, the relative ordering of the implicit sources is important for the sign determination. Diagrammatically we fix the sign by requiring that all the implicit sources lie to the right of the pulled legs:

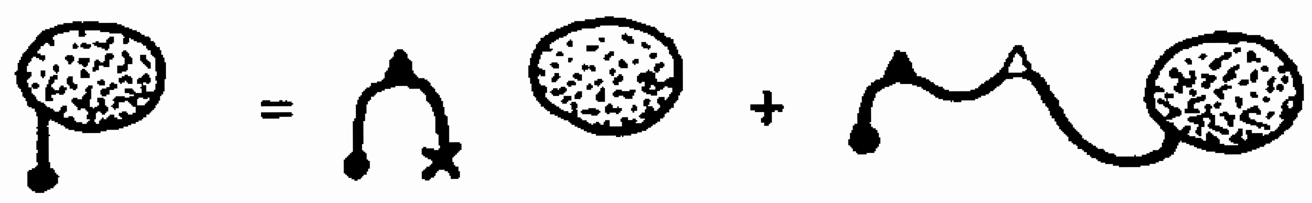
$$\frac{d}{d\eta_i} Z[\eta] =$$  $$\qquad (4.9)$$

<u>Exercise 4.B.1</u> <u>Fermionic loops</u>. (This exercise is a convoluted attempt to prove the minus sign rule for fermions by diagrammatic means.) The simplest interacting fermionic field theory has only a bilinear interaction term:

$$S_I[\psi] = \frac{1}{2}\psi_i \not{A}_{ji}\psi_j$$

$$\not{A}_{ij} =$$  $$= -\not{A}_{ji}\ .$$

Here $\not{A}$ could be an external background photon field $\not{A}_{ij} = gA_\mu(\gamma^\mu)_{ij}$, as in (3.27). The DS equations corresponding to (4.4) are



$$\frac{d}{d\eta_i}Z[\eta] = \Delta_{ij}\left(\eta_j + \not{A}_{kj}\frac{d}{d\eta_k}\right)Z[\eta]\ . \qquad (4.10a)$$

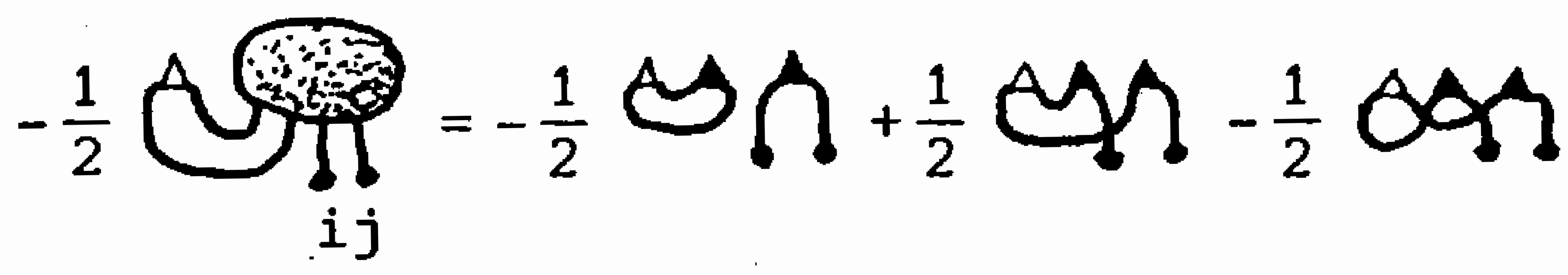
Construct the DS equation for pulling out a "photon" $\not{A}$. This can be done by differentiating $Z[\eta]$ with respect to the coupling constant; a 2-leg vertex gets pulled out. Pull the first fermion leg. It either ends in the second leg, on a source, or on a 2-leg vertex:

$$g\frac{d}{dg}Z[\eta] = -\frac{1}{2}$$ 

$$= \frac{1}{2}\Big($$  $$\Big)$$

$$= \frac{1}{2}\left(-\mathrm{tr}\not{A}\Delta + \eta_i\Delta_{ij}\not{A}_{jk}\frac{d}{d\eta_k} + (\not{A}\Delta\not{A})_{ij}\frac{d}{d\eta_j}\frac{d}{d\eta_i}\right)Z[\eta] \qquad (4.10b)$$

According to our convention (4.9) all implicit sources lie to the right of the explicit legs. The real trick consists of getting the signs straight. The relative sign between the first and second term is due to the antisymmetry of fermionic Green functions. The overall sign is fixed by requiring consistency with the DS equations (4.10a). For example, if we substitute the 4-leg free fermionic Green function (4.4) into the above, we obtain



$$= -\frac{1}{2}(\mathrm{tr}\Delta\not\lambda)\Delta_{ij} + \Delta_{ij}\not\lambda_{jk}\Delta_{kj} \quad .$$

The sign of the connected term must be consistent with the expansion (4.11):

Show by iterating (4.10b) that

$$W[\eta] = \frac{1}{2}\mathrm{tr}\,\ell n(1-\Delta\not\lambda) + \frac{1}{2}\eta_i\left(\frac{1}{\Delta^{-1}-\not\lambda}\right)_{ij}\eta_j \quad . \tag{4.11}$$

Compare with (3.25). The difference between the bosonic and the fermionic theories is that each fermionic loop carries a factor -1.

Exercise 4.B.2 Derive the relations between the full, connected and 1PI fermionic generating functionals. Write down the Dyson-Schwinger equations such as

$$\left(\frac{dS}{d\psi_i}\left[\frac{d}{d\eta}\right] + \eta_i\right)Z[\eta] = 0 \tag{4.12}$$

without getting confused about fermionic signs.

C. Fermion arrows

In the literature, fermionic generating functionals are never defined in terms of a single source, as in (4.7). We have introduced them in this way to parallel the bosonic formalism. However, usually a pair of sources is used; one for fermions, and one for antifermions. We shall now rewrite the fermionic generating functionals in this more conventional form.

We start by considering the most trivial fermionic theory; we take the range of the collective index to be $i = 1,2$. The propagator is a (2×2) antisymmetric matrix:

$$\Delta = \begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}$$

and the action (2.13) takes the form

$$S[\psi] = -\frac{1}{2}\psi_i\Delta^{-1}_{ij}\psi_j = -\frac{1}{\lambda}\psi_1\psi_2 \quad .$$

The (2×2) matrix Δ has eigenvalues $\pm i\lambda$. We can eliminate this matrix by replacing ψ_1, ψ_2 by

$$\psi = \frac{1}{\sqrt{2}}\left(\psi_1 + \psi_2\right), \bar{\psi} = \frac{1}{\sqrt{2}}\left(\psi_1 - \psi_2\right) ,$$

(this is reminiscent of the introduction of charged bosonic fields, equation (3.12)). The propagator is now just a number:

$$S[\bar{\psi},\psi] = - \bar{\psi}\,\frac{1}{\lambda}\,\psi \; .$$

Matrix Δ is invertible only if $\mathrm{Det}\,\Delta \neq 0$. For an antisymmetric matrix this is possible only in even dimensions. A real antisymmetric $(2m\times 2m)$ matrix Δ_{ij} can always be brought to form

$$G^{-1}\Delta G = \begin{bmatrix} 0 & -\lambda_1 & & & & & & \\ \lambda_1 & 0 & & & & & & \\ & & 0 & -\lambda_2 & & & & \\ & & \lambda_2 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & & 0 & -\lambda_m \\ & & & & & & \lambda_m & 0 \end{bmatrix} \qquad (4.13)$$

by means of a sympletic rotation $G\varepsilon Sp(2m)$. (This is the fermionic analogue of the diagonalization which leads to (3.6).) Defining

$$\psi_i = \frac{1}{\sqrt{2}}\left(\psi_{2i-1} + \psi_{2i}\right), \; \psi^i = \frac{1}{\sqrt{2}}\left(\psi_{2i-1} - \psi_{2i}\right) \qquad (4.14)$$

we can write the free action as

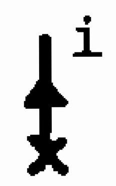
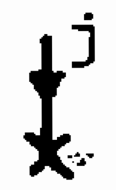
$$S[\bar{\psi},\psi] = -\bar{\psi}\Delta^{-1}\psi = - \psi^i \Delta_i^{-1j}\psi_j \quad , \qquad (4.15)$$

where the propagator is now an $(m\times m)$ matrix which in the diagonalized form looks like

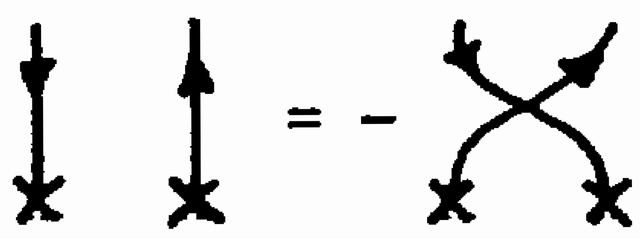
$$\Delta_i^j = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_m \end{bmatrix} .$$

In this way a 2m-dimensional fermionic field ψ_i can always be replaced by a pair of m-dimensional fields $\bar{\psi},\psi$. Diagrammatically we distinguish the upper and the lower indices by drawing arrows flowing <u>away</u> from upper indices and <u>into</u> the lower indices:

$\Delta_i^j =$ 

$\eta_i =$  , $\eta^j =$  . (4.16)

One advantage of the fermion-antifermion formalism is that the antisymmetric propagator (4.13) is replaced by (4.16) which carries no funny signs. However, it still follows from the definition (4.14) that the fermion and the antifermion fields and sources are anticommuting:



$$\eta^i\eta_j = -\eta_j\eta^i \quad . \tag{4.17}$$

The fermionic generating functionals are now a double series in terms of the fermion, antifermion sources:

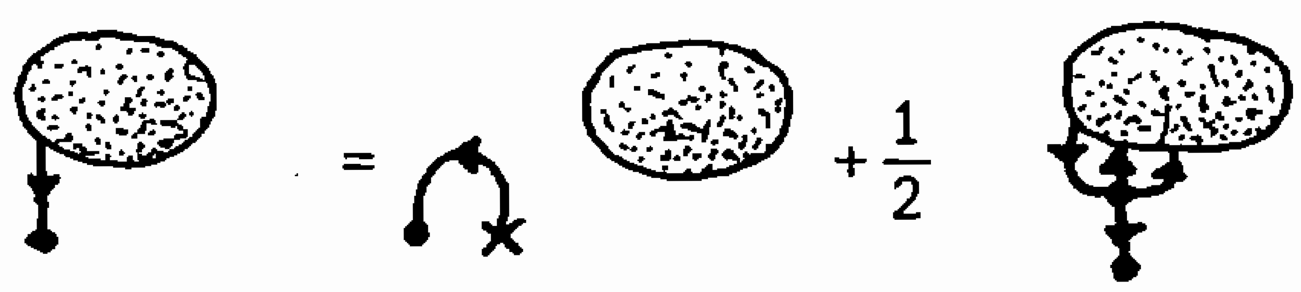
$$Z[\bar{\eta},\eta] = \sum_{m,n} Z_{\underbrace{ijk\ldots\ell}_{m\text{ out-legs}}}^{\overbrace{i'j'\ldots k'}^{n\text{ in-legs}}} \frac{\eta_{k'}\ldots\eta_{j'}\eta_{i'}}{n!}\,\frac{\eta^{\ell}\ldots\eta^{j}\eta^{i}}{m!} \quad . \tag{4.18}$$

<u>Exercise 4.C.1</u> <u>Fermionic loops</u>. Show that the connected generating functional for fermion propagation in a background "photon" field is given by:

$$W[\bar{\eta},\eta] = \quad \ln \mathrm{tr}(1-\Delta A\!\!\!/) + \bar{\eta}\frac{1}{\Delta^{-1} - A\!\!\!/}\eta \ . \tag{4.19}$$

Compare with the bosonic case (3.28). The difference between the bosonic and the fermionic theories is that each fermionic loop carries a factor -1.

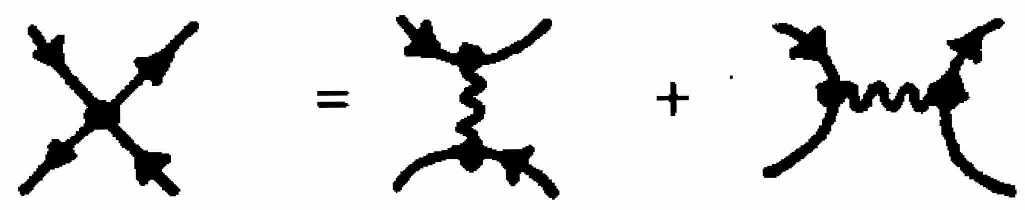
<u>Exercise 4.C.2</u> <u>Dyson-Schwinger equations</u>. The fermionic $(\bar{\psi}\psi)^2$ theory DS equations for full Green functions are given diagrammatically by



Show that the DS equations for directed fermions can be written as

$$\left(\frac{dS}{d\psi_i}\left[\frac{d}{d\eta}, \frac{d}{d\bar{\eta}}\right] + \eta^i\right) Z[\bar{\eta},\eta] = 0 \ . \tag{4.20}$$

Exercise 4.C.3 QED DS equations. The four vertex in the preceeding exercise could be a phenomenological approximation to a boson exchange (Fermi theory of weak interactions is of this type)



(is this consistent with fermionic symmetry?). Add a boson propagator to the theory and write the boson and fermion DS equations for this theory.

D. Fermionic path integrals

We have seen in chapter 3 that a lot can be gained by defining a "Fourier" transform which diagonalizes the differential operators:

$$\frac{d}{d\eta_i} Z[\eta] \rightarrow \psi_i \tilde{Z}[\psi] \ . \tag{4.21}$$

For fermions the derivatives anticommute (4.8) so ψ_i have to be anticommuting numbers. Let us blindly imitate the bosonic case and write down

$$Z[\eta] = \int [d\psi] e^{\eta_i \psi_i} \tilde{Z}[\psi] \ .$$

What is this "integral"? Consider first the one-dimensional case. The left-hand side must be independent of ψ and, in particular, invariant under translations $\psi \rightarrow \psi + \theta$:

$$\int [d\psi]\psi = \int [d\psi](\psi + \theta) \ .$$

This works only if

$$\int [d\psi] = 0$$

$$\int [d\psi]\psi \neq 0 \ .$$

We take $\int [d\psi]\psi = 1$ (just a normalization convention). As $\psi^2 = \psi^3 = \ldots = 0$, there are no other integrals to be evaluated. The integration operation must be anticommutative because $\psi\theta = -\theta\psi$ implies

$$\int [d\psi]\theta\psi = -\theta\int [d\psi]\psi = -\left[\int [d\psi]\psi\right]\theta \ .$$

The generalization to many dimensions is

$$\int [d\psi_i]\psi_j = \delta_{ij} \ . \tag{4.22}$$

Curiously, the fermionic "integration" is indistinguishable from the fermionic "differentiation" (4.8). It is really no integration at all; it is simply an operational rule which implements the desired diagonalization (4.21):

$$\begin{aligned} \frac{d}{d\eta_i} Z[\eta] &= \int [d\psi]\frac{de^{\eta_j\psi_j}}{d\eta_i} \tilde{Z}[\psi] \quad - \frac{d}{d\eta} Z[\eta] \\ &= \int [d\psi]\psi_i\tilde{Z}[\psi] \ , \end{aligned} \tag{4.23}$$

(as usual, we assume that the number of fermionic dimensions is even). Now, just as in the bosonic case (3.4), we can compute $\tilde{Z}[\psi]$ from (4.20) by solving the fermionic Dyson-Schwinger equation:

$$Z[\eta] = \int [d\psi]e^{S[\psi] + \eta_i\psi_i} \ . \tag{4.24}$$

This is the path integral representation for the fermionic Green functions.

<u>Exercise 4.D.1</u> Can you think of a simple argument which will give the correct $i\varepsilon$ prescription for fermionic propagators, analogous to (3.10) for the bosonic theory?

<u>Exercise 4.D.2</u> Check (4.23).

E. Fermionic determinants

The simplest fermionic analogue to the bosonic gaussian integral (3.5) is the 2-dimensional integral

$$\int[d\psi_1 d\psi_2]e^{-\frac{1}{2}\psi_i\Delta^{-1}_{ij}\psi_j} = \int[d\psi_1 d\psi_2](1-\frac{1}{\lambda}\psi_1\psi_2)$$

$$= \frac{1}{\lambda} = \mathrm{Det}\,\Delta^{-\frac{1}{2}} \quad , \tag{4.25}$$

where

$$\Delta_{ij} = \begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix} .$$

In odd dimensions such integrals always vanish, as at least one $\int[d\psi_i]$ is unmatched. In even dimensions

$$\int[d\psi]e^{-\frac{1}{2}\psi_i\Delta^{-1}_{ij}\psi_j} = \mathrm{Det}\,\Delta^{-\frac{1}{2}} \quad . \tag{4.26}$$

> Derivation (analogous to (3.6)): $\Delta_{ij} = -\Delta_{ji}$, hence there exists a sympletic rotation G such that Δ_{ij} can be brought to form (4.13). Sympletic rotations are volume preserving, so $d(G\psi) = d\psi$. This rotation reduces the 2m dimensional integral to a product of m two-dimensional integrals (4.25): the result is
>
> $$\prod_{i=1}^{m}\frac{1}{\lambda_i} = \mathrm{Det}\,\Delta^{-\frac{1}{2}} .$$ QED.

The important thing to note is that the fermionic "gaussian" integral yields inverse determinant, in contrast to the bosonic integral (3.6). If you repeat the saddlepoint analysis of sect. 3.F and use $\ln(\det M) = \mathrm{tr}(\ln M)$ rule (3.23), you will find that in the fermionic case the effective action (3.25) is given by

$$\Gamma[\psi^c] = S[\psi^c] - \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}\mathrm{tr}(\Delta\gamma[\psi^c])^K \quad . \tag{4.27}$$

As we have already shown diagrammatically in exercises 4.B.1 and 4.C.1, each fermion loop carries a factor -1.

Exercise 4.E.1 Introduce a source term $\eta_i\psi_i$ in (4.26) and compute the generating functional (cf. (3.7)) for the free fermionic field theory.

Exercise 4.E.2 Show that for directed fermions, sect.4.C, the fermionic gaussian integral is given by

$$\int[d\psi d\bar{\psi}]e^{-\bar{\psi}\Delta^{-1}\psi} = \frac{1}{\mathrm{Det}\,\Delta} \quad . \tag{4.28}$$

F. Fermionic jacobians

The only possible redefinition of a one-dimensional fermionic integration variable is

$$\psi \to \psi' = a\psi + \theta \ .$$

The jacobian $d\psi = J d\psi'$ must be such that the integration rule (4.22) is preserved

$$1 = \int [d\psi]\psi = \int [d\psi'] J \frac{\psi' - \theta}{a} = \frac{J}{a} \ .$$

Hence the jacobian is $J = d\psi'/d\psi$, the <u>inverse</u> of the bosonic jacobian. That is easy to understand if one remembers that the fermionic "integration" is the same thing as the fermionic differentiation:

$$\int [d\psi] = \frac{d}{d\psi_1}\frac{d}{d\psi_2}\cdots\frac{d}{d\psi_{2m}}$$
$$= \left(\frac{d\psi'_i}{d\psi_1}\frac{d\psi'_j}{d\psi_2}\cdots\frac{d\psi'_k}{d\psi_{2m}}\right)\frac{d}{d\psi'_i}\frac{d}{d\psi'_j}\cdots\frac{d}{d\psi'_k} \ .$$

As the fermionic differentiations anticommute, the term in the brackets is fully antisymmetric; the determinant. The jacobian in 2m dimensions is therefore

$$\int [d\psi] = \int [d\psi'] \det\left(\frac{d\psi'_i}{d\psi_j}\right) \ , \tag{4.29}$$

the inverse of a bosonic jacobian.

<u>Exercise 4.F.1</u> <u>A trivial supersymmetry</u>. Take one bose and two Fermi dimensions. Using $\det\Delta/\det\Delta = 1$, we can write

$$1 = \int [dA d\bar{\omega} d\omega] e^{-\frac{A^2}{2\lambda} - \bar{\omega}\frac{1}{\sqrt{\lambda}}\omega} \ .$$

It is very easy to find a supersymmetry of this action. A shift

$$A \to A + \varepsilon\sqrt{\lambda}\omega \quad , \qquad \varepsilon \text{ fermionic,}$$

produces an extra term in the action: $-A\varepsilon\omega/\sqrt{\lambda}$. This can be compensated by a shift of the antifermionic field

$$\bar{\omega} \to \bar{\omega} - \varepsilon A \ .$$

The action $S[A,\bar{\omega},\omega]$ of this free field theory is therefore invariant under supersymmetric (Fermi-bose mixing) transformations

$$A \to A + \varepsilon\sqrt{\lambda}\omega$$
$$\bar{\omega} \to \bar{\omega} - \varepsilon A \quad . \tag{4.30}$$

Add sources

$$Z_0[J,\bar{\eta},\eta] = \int [dA d\bar{\omega} d\omega] e^{S[A,\bar{\omega},\omega]+JA+\bar{\eta}\omega+\bar{\omega}\eta}$$

and show that the supersymmetry induces a Ward identity of type (3.34). Verify diagrammatically that the identity is satisfied. This is quite trivial, and still, the QED Ward identities amount to no more than this. In that case A is the photon field, $\sqrt{\lambda}$ longitudinal insertion k^μ, and ω the QED ghost which nobody cares about because it always decouples.
The main lesson of this exercise is this: if we (1) create fake boson degrees of freedom and (2) remove them by ghosts, the theory might have a hidden supersymmetry.

G. Summary

Fermions (or Grassmann numbers) are tricks for manipulating antisymmetric Green functions. Green functions are still ordinary numbers (real for statistical mechanics, complex for quantum mechanics), and there is no mystique in computing them (only tedium). The physical content of fermions is that they offer a way of imposing constraints. One such constraint is Pauli principle - electrons are fermions. The QCD ghosts which we will construct in chapter 6 are another example: they eat up the unphysical longitudinal gluon degrees of freedom. Physically, fermions are to be counted as negative degrees of freedom (fermion loops carry minus signs) which cancel the unphysical bose degrees of freedom.

Fermionic Green functions are antisymmetric under interchange of indices. The fermionic sources and fields anticommute;

$$\eta_i\eta_j = -\eta_j\eta_i \quad , \qquad \psi_i\psi_j = -\psi_j\psi_i \quad ,$$

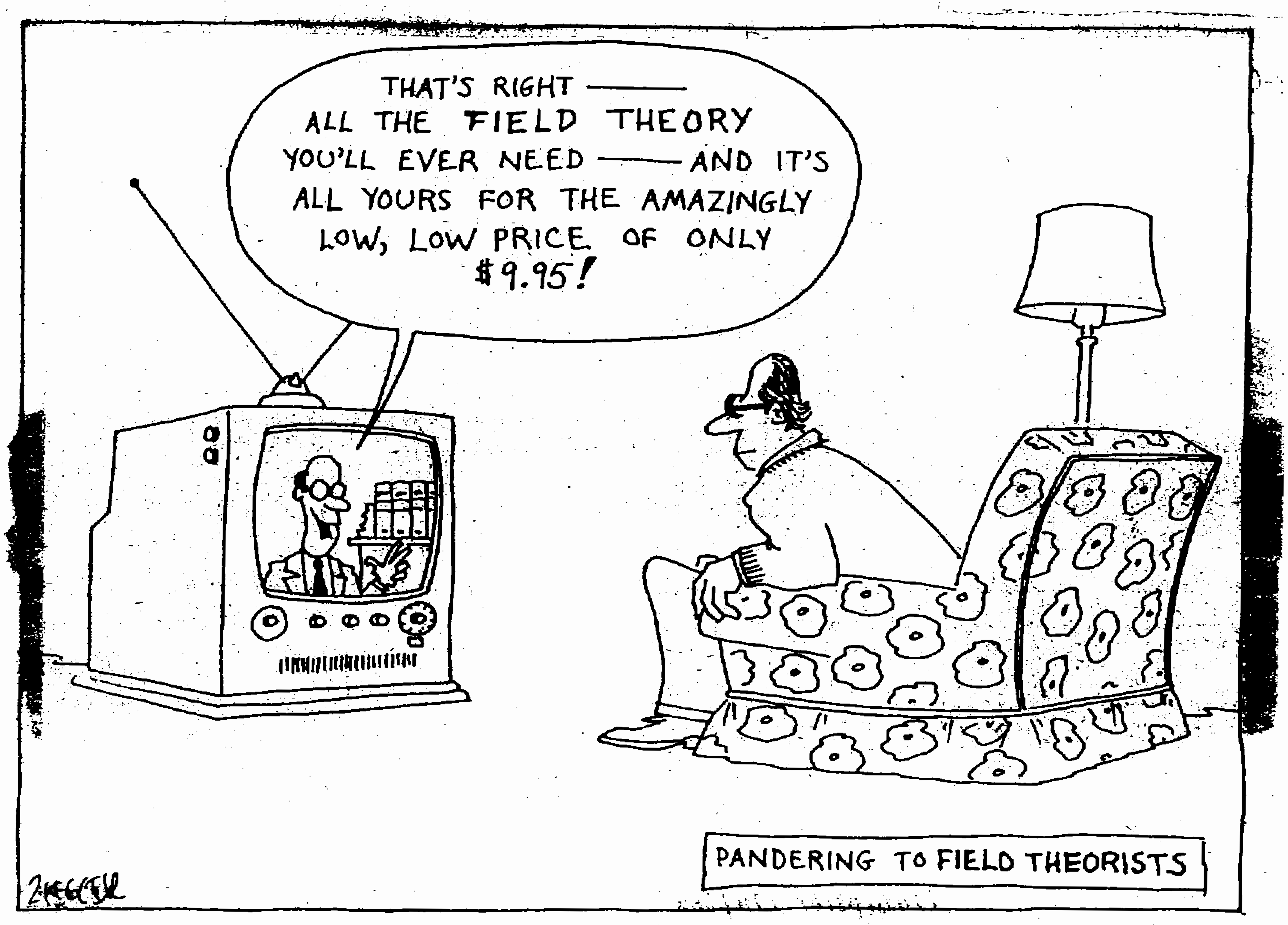
$$\frac{d}{d\eta_i}\eta_j = \delta_{ij} - \eta_j\frac{d}{d\eta_i} \quad ,$$

$$\frac{d}{d\eta_i}\frac{d}{d\eta_j} = -\frac{d}{d\eta_j}\frac{d}{d\eta_i} \quad .$$

The fermionic integrals are defined by

$$\int [d\psi_i] d\psi_j = \delta_{ij} \quad .$$

The entire machinery developed for bose fields applies to Fermi fields, modulo few irrelevant sign confusions and one relevant sign; factor -1 for each fermionic loop.



PANDERING TO FIELD THEORISTS